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Kripke Completeness for Intermediate Logics

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

The thesis aims to investigate some problems related to *superintuitionistic propositional intermediate logics*, that is, sets of propositional formulas closed with respect to modus ponens and substitutions (of propositional variables with formulas) that lie between the (propositional) logic **Int** and the (propositional) classical logic **Cl**. There is a continuum of such logics L, and we can characterize them by syntactical or semantical tools. In the former case, we give a sequence of formulas (axioms) from which, by means of an effective set of rules, we can derive all the formulas of L; in the latter case, we provide a semantics in which the valid formulas are exactly those in L. The most popular semantics for intermediate logics are the algebraic semantics and the Kripke frames semantics. Algebraic semantics has a privilegiated role, since every intermediate logic L admits an algebraic semantic characterization. Kripke semantics was firstly introduced in the study of modal systems and successively it was revealed a good tool in describing intermediate logics. Indeed, it seems to be very manageable and very suitable to treat many important theoretical problems which arise in logic (such as the problems faced in the thesis). A drawback is that, differently from algebraic semantics, not all the intermediate logics can be characterized by kripkean semantics, and in literature some examples of such logics are presented (see for instance [31, 32]). Nevertheless, many interesting logics, such as the ones presented in the thesis (see Chapter 2), can be described by frames semantics (we recall however that these remarks do not hold for predicate logics, where the use of kripkean semantics is rather problematic and many incompleteness phenomena arise). This relationship between modal logics and intermediate logics is not casual, since there is a deep analogy between them; indeed the main notions concerning intermediate logics actually derives from modal logics. One can observe that some central notions, such as completeness, strong completeness, canonicity, are well established in the literature of modal logics (see for instance [6]). This also suggests that many ideas and results of our research (for instance, the criteria for canonicity and strong completeness) can be translated in the modal framework without great effort.

In this thesis we confine ourselves to treat intermediate propositional logics and our reference semantics is the kripkean one, hence we begin by recalling what we mean by Kripke frame semantics. A Kripke frame is a partial order, that is a set of points (states of the frame) equipped with a partial ordering relation; a Kripke model is obtained by defining an interpretation of the language (more precisely, of the propositional variables of the language) on a frame. When all the models $\underline{K}$ based on a frame $\underline{P}$ are model of L (i.e., all the formulas of L are valid in all the points of $\underline{K}$), we say that $\underline{P}$ is a *frame for* L. Given a nonempty class $\mathscr{F}$ of frames, the set of formulas $\mathscr{L}(\mathscr{F})$ valid in all models based on the frames of $\mathscr{F}$ is actually an intermediate logic (for a precise definition of these notions see Chapter 1 or the textbook [4], which is the main reference as regards the general notions about intermediate logics).

A major concern in logic is the relationship between the syntactical apparatus of a logic and its semantical counterpart. This is the well known *completeness* problem which can be stated in the following terms: given a logic L, is there a class of frames $\mathscr{F}$ such that $\mathscr{L}(\mathscr{F}) = L$? A stronger notion is that of *strong completeness* which concerns the relationship between the syntactical notion of "derivability in L of a set of formulas Δ from a set of formulas Γ" ($\Gamma \vdash_L \Delta$) and the semantic notion of "logical consequence of Δ from Γ, with respect to the class $\mathscr{F}$ of all frames for L" ($\Gamma \models_{\mathscr{F}} \Delta$). In this case, the question is: is it true that $\Gamma \vdash_L \Delta$ iff $\Gamma \models_{\mathscr{F}} \Delta$? As it happens for modal logics, in some cases the proof of strong completeness of a logic can be carried out by means of simple tools; this happens, for instance, for *canonical logics*. We recall that the *canonical model* of a logic L is, in some sense, the biggest model of L, since it contains all the saturated sets (that is, consistent sets of formulas which satisfy some closure properties) which include L. When the frame of the canonical model is a frame for L, we say that L is canonical; as an immediate consequence of the definition, one gets that canonical logics are complete and even strongly complete and this justifies the relevance of this notion. In this connection, we recall a difficult open question: does the class of canonical logics coincide with the class of strongly complete logics? As far as we know, there are not convincing arguments in favour of either option, we also think that the tools developed in the thesis are not useful to solve the question (as a matter of fact, every time we find a counterexample to show that a logic L is not canonical, we are able, with slight modifications, to disprove even the non strong completeness of L). The proof of canonicity consists in checking that the canonical model satisfies certain properties; in general, such a proof is carried out by ad hoc arguments which depend on the logic in hand. One can also observe that such proofs have not the same structure: sometimes the proof goes very smoothly, since only elementary properties of canonical models are used; in other cases the proof is more involved, since strong properties of canonical models (such as the fullness) must be used. A merit of [15] is to introduce a classification of canonicity which takes into account these different patterns involved in canonicity proofs, distinguishing some *degrees of canonicity* and also stating some criteria for classifying the logics. This approach is quite original in literature, since it is a first contribute devoted to treat in a systematic way these central notions, and this is the starting point of our research. First of all, we propose and refine the classification of canonicity: we single out, as the simplest case, the *hypercanonicity* (distinguishing three subcases) and the *extensive canonicity*, while we consider non elementary the other cases of canonicity (see Chapter 3 for such a refined classification, where many examples are presented to separate these various subcases of canonicity). Then (Chapter 4), integrating [15], we state criteria for hypercanonicity, extensive canonicity and strong completeness. Even if our criteria are formally similar to the ones in [15], the techniques and the arguments we use are different: while in the quoted paper the authors use algebraic-categorical tools, we directly act on kripkean semantics, using techniques more inspired to the classical Model Theory. The most interesting application of these results is to the class of *logics in one variables*, that is, superintuitionistic logics having as extra axiom a formula containing only one propositional variable. This family of logics is well studied and characterized in literature. Nishimura first [30] introduced an effective enumeration of non intuitionistically equivalent formulas F_n in one variable; the logics in one variable are then obtained by adding to **Int** any formula F_n when it makes sense (that is, F_n is classically valid) and taking care into the cases in which different formulas yield the same logic. The first careful analysis of such logics is given by Anderson in [1], where the problem of disjunction property is also treated (for more details, see [3, 4]). Further, an important result has been obtained by Sobolev in [34], who has shown that all the logics in one variable have the finite model property, hence they are decidable and admit a semantics in terms of Kripke frames (see also Chapter 2). As regards the classification of these logics with respect to strong completeness, it is not difficult to show that four of them are canonical (hence strongly complete). In [15] appeared the first significative and systematic result and it is proved that:

- All the logic axiomatized by axioms in one variable, except four of them, are not canonical.

We observe that, before this paper, the only known result was the non canonicity of the *Scott logic* **St** ([33]). We point out that these results are not trivial, since one has to deal with "very big" countermodels. While there is not any difficulty in treating with finite models (see Chapter 1), many problems may arise with the infinite ones. The ideas beyond these criteria is to overcome these troubles by building an infinite model as a sort of *limit* of an infinite sequence (chain) of finite models, in such a way that, passing from the finite models of the chain to the infinite limit, many properties are preserved. Here, we propose an improvement of the previous result, taking also into account the logics in one variable in *finite slices*. More precisely, let us denote with L_n the n^{th} logic in one variable of our enumeration, and let us denote with $L_{n,h}$ the logic characterized by the frames of L_n with depth at most h (note that L_n is properly contained in $L_{n,h}$). Then, we show that:

- For each n such that L_n is not strongly complete, there is h such that:
    - $L_{n,j}$ is canonical for $j \leq h$;
    - $L_{n,j}$ is not strongly complete for $j > h$.

The only exception is the case of the *Anti-Scott logic* **Ast**([9]); indeed, even if such a logic is not strongly complete, all its finite slices are canonical (the proof of this fact is not trivial). We also give other minor applications of these criteria to the well known *Medvedev logic* ([26, 27]) and to the so called *logic of rhombuses* ([23]). This concludes our systematic exposition about canonicity and strong completeness.

Even if a logic L is not canonical, one can state the completeness of L by showing that L satisfies a property weaker than canonicity, called *ω-canonicity* (or *weak canonicity*). We remark that for logics which are not even ω-canonical the use of sophisticated tools, such as filtration techniques, becomes unavoidable. Incidentally, we point out that filtration techniques are in general mandatory if one wants to prove stronger properties of a logic, such as the finite model property (closely related to the decidability of a logic); however, in the thesis we will not treat this subject.

The notions of *ω-canonicity*, *extensive ω-canonicity*, *strong ω-canonicity* derive from a weakening of the corresponding definitions explained above (in practice, we consider only languages $\mathscr{L}_V$ generated by finite sets of propositional variables V). As one expects, strong completeness implies strong ω-completeness (and the same holds for the other notions), while the converse is not true, as it will be shown by many examples. Similarly, the ω-canonical model with respect to some logic L and to some finite set V is defined as the canonical model of L, where we identify the points which cannot be distinguished by formulas of $\mathscr{L}_V$. The ω-canonical models satisfy some interesting properties: they have finitely many final points (while the canonical models may have uncountable many final points) and they have a sort of "filter property". More precisely, if a point α has infinite depth, then there is a point β such that β has infinite depth and, for all $\delta > \beta$, δ has finite depth (in particular, β has no immediate successors); moreover, for every $\delta_1, \delta_2 > \beta$, there is γ such that $\beta < \gamma$, $\gamma < \delta_1$ and $\gamma < \delta_2$, that is, β behaves as a filter. A report of these properties (which are not trivial and are unpublished) can be found in Appendix A. Owing to these nice properties, in general the proof of completeness is easier if one start from an ω-canonical model (and in some cases one can also immediately prove the finite model property).

The most original part of the thesis is the one regarding the study of these weak notions, which are scarcely investigated in literature. The methods here developed (see Chapter 5) are completely original and are different (and perhaps more powerful) from the ones explained in [15]. These ideas are quite new in literature and, as far as we know, no examples of non strongly ω-complete logics were known. In [15] it is showed that all the *Gabbay de-Jongh logics* (see [13] and see Chapter 2, where are called *logics of finite branching*) are not strongly ω-complete. The tools used in the proof essentially derive from a relativisation of the techniques developed for canonicity. Here we follow a different approach. Indeed, instead of using sequences of models to built our (very complex!) counterexamples, we directly define this big models, and then we check that they satisfy the properties we need. This requires the introduction of new notions, such as the ones of *V-grade* (of a point of an ω-canonical model), of *V-sequence* of points, of *limit* of a V-sequence; these notions naturally yield to the formulation of necessary and sufficient conditions for the (well) V-separability and the V-fullness of models. In general, these tools allow to be more loose in the definition of countermodels, if compared with the method of the chains, where many constraints are imposed.

The main application regards the logics in one variable: indeed, we considerably improve the above result and we show that:

- All the logics axiomatized by formulas in one variable, except eight of them, are not strongly ω-complete.

This is perhaps the most important result of the thesis since it is a definitive result about the classification of such a family of logics.

The thesis is organized as follows. In Chapter 1 we introduce the basic notions, while in Chapter 2 we present some intermediate logics. In Chapter 3 we begin the analysis of the main notions investigated in the thesis, namely canonicity, extensive canonicity, strong completeness, ω-canonicity, extensive ω-canonicity, strong ω-completeness and other related ones. A careful analysis of these concepts is developed, with many examples. In Chapter 4 some original techniques for the study of canonicity, extensive canonicity and strong completeness of intermediate logics are presented; it is also proposed a refined classification of the logics axiomatized by formulas in one variable. Other applications are given for the Medvedev logic and the logic of rhombuses. Chapter 5 is the most original part of the thesis. Here we develop some techniques for the analysis of ω-canonicity, extensive ω-canonicity and strong ω-completeness. As an application, it is proved the above mentioned result about the logics with extra axiom in one variable. Appendix A contains a deeper study of ω-canonicity; in particular, some interesting (and non trivial) properties of ω-canonical models are proved. Appendix B summarizes the main results of the thesis.

1.2 Preliminary definitions

As usual, a *(Kripke) frame* is a pair $\underline{P} = \langle P, \leq \rangle$ consisting of a nonempty set P and a partial order $\leq$ on P, i.e., $\underline{P}$ is a partially ordered set (*poset*). The elements of P are called the *points* of the frame $\underline{P}$ and $\alpha \leq \beta$ is read as "β is accessible from α" or "α sees β". We write $\alpha < \beta$ to mean that $\alpha \leq \beta$ and $\alpha \neq \beta$; we also use the

notations $\beta \geq \alpha$ and $\beta > \alpha$ as a synonymous of $\alpha \leq \beta$ and $\alpha < \beta$ respectively. A *subframe* of $\underline{P}$ is a frame $\underline{P}' = \langle P', \leq' \rangle$ obtained by considering a subset P' of P and the restriction $\leq'$ of $\leq$ to P'; the subframe is said to be a *generated subframe* iff P' is upward closed. If α is a point of $\underline{P}$, the *cone* $\underline{P}_\alpha$ of $\underline{P}$ is the generated subframe of $\underline{P}$ obtained by considering α and all the points accessible from α.
A point β is an *immediate successor* of α if $\alpha < \beta$ and, for all points γ of $\underline{P}$ such that $\alpha \leq \gamma \leq \beta$, we have either $\gamma = \alpha$ or $\gamma = \beta$.
A *final point* of a frame $\underline{P} = \langle P, \leq \rangle$ is a maximal point of $\underline{P}$; $\mathrm{Fin}(\alpha)$ denotes the set of all the final points accessible from α. We say that $\underline{P}$ has *enough final points* iff, for every $\alpha \in P$, $\mathrm{Fin}(\alpha) \neq \emptyset$. We say that α has *depth* n (and we write $\mathrm{depth}(\alpha) = n$) if n is the maximum length of a chain of points starting from α (namely, there is a sequence of n points of $\underline{P}$ $\alpha_1 \equiv \alpha < \alpha_2 < \cdots < \alpha_n$ and any other sequence of this kind contains at most n points). Clearly, a final point has depth 1. The depth of a frame $\underline{P}$ is the maximum between the depths of the points of $\underline{P}$.
We use the notation $\underline{P} = \langle P, \leq, \rho \rangle$ to indicate a frame $\underline{P}$ with *root* ρ, where as usual the root is the minimal point of $\underline{P}$; in this case, we say that $\underline{P}$ is a *rooted* frame.
In the sequel, we will assume to fix a *propositional language* $\mathscr{L}_\mathscr{V}$, containing the *propositional connectives* $\wedge, \vee, \rightarrow, \neg$ and a countable set of *propositional variables* $\mathscr{V}$. The *formulas* of $\mathscr{L}_\mathscr{V}$ are defined in the usual way. Given a formula A, $\mathrm{Var}(A)$ denotes the (finite) set of propositional variables occurring in A. If $\mathrm{Var}(A) \subseteq V$, where V is a set of propositional variables such that $V \subseteq \mathscr{V}$, we say that A is a V-formula.
A *substitution* σ is a map from $\mathscr{V}$ to the formulas of $\mathscr{L}_\mathscr{V}$; σA denotes the formula obtained by replacing every propositional variable p occurring in A with the formula σp.
Let $\underline{P} = \langle P, \leq \rangle$ be a frame; a *Kripke model* $\underline{K} = \langle P, \leq, \Vdash \rangle$ is obtained by defining a *forcing relation* $\Vdash$ between any point α of $\underline{P}$ and any propositional variable p of $\mathscr{V}$, in such a way that:

$$\alpha \Vdash p \quad \text{and} \quad \alpha \leq \beta \implies \beta \Vdash p.$$

The notation "$\alpha \Vdash p$" (α forces p) means that the pair $\langle \alpha, p \rangle$ belongs to the forcing relation; if this is not true, we say that "$\alpha \nVdash p$". When $\underline{K} = \langle P, \leq, \Vdash \rangle$, we say that $\underline{K}$ is *based* on the frame $\underline{P} = \langle P, \leq \rangle$ and that $\underline{P}$ is the (underlying) frame of $\underline{K}$. The forcing relation is extended to all the formulas of $\mathscr{L}_\mathscr{V}$ in the usual way. More precisely, let $\underline{P} = \langle P, \leq \rangle$ be a frame and let us suppose that the forcing relation $\Vdash$ has been already defined on the propositional variables; then we define the relation $\alpha \Vdash A$, for every $\alpha \in P$ and every formula A of $\mathscr{L}_\mathscr{V}$, inductively on the structure of A, as follows:

- $A = B \wedge C$, and $\alpha \Vdash B$ and $\alpha \Vdash C$;
- $A = B \vee C$, and either $\alpha \Vdash B$ or $\alpha \Vdash C$;
- $A = B \rightarrow C$, and, for every $\beta \in P$, if $\alpha \leq \beta$ and $\beta \Vdash B$, then $\beta \Vdash C$;

- $A = \neg B$, and, for every $\beta \in P$, if $\alpha \leq \beta$ then $\beta \nVdash B$.

It is not difficult to prove that the above conservation property of the forcing holds for all the formulas A of $\mathscr{L}_{\mathscr{V}}$, that is:

$$\alpha \Vdash A \quad \text{and} \quad \alpha \leq \beta \implies \beta \Vdash A.$$

Submodels, generated submodels and cones of models are defined similarly to subframes, generated subframes and cones of frames.
Given a model $\underline{K} = \langle P, \leq, \Vdash \rangle$ and $\alpha \in P$, $\Gamma_{\underline{K}}(\alpha)$ (or simply $\Gamma(\alpha)$ if the context is clear) denotes the set of the formulas forced in α. If V is a set of propositional variables, $\Gamma^V_{\underline{K}}(\alpha)$ (or simply $\Gamma^V(\alpha)$ if the context is clear) denotes the set of V-formulas forced in α.
We say that a formula A is *valid* in $\underline{K}$ (and we write $\underline{K} \models A$) iff $\alpha \Vdash A$ for all $\alpha \in P$; we say that a set of formulas Δ is *valid* in $\underline{K}$ (and we write $\underline{K} \models \Delta$) iff $\underline{K} \models A$ for every $A \in \Delta$. In this case we also say that $\underline{K}$ is a model of Δ.
We say that a formula A is *valid* in a frame $\underline{P}$ (and we write $\underline{P} \models A$) iff $\underline{K} \models A$ for every Kripke model $\underline{K}$ based on $\underline{P}$; $\underline{P} \models \Delta$ iff $\underline{P} \models A$ for every $A \in \Delta$.
Let Γ and Δ be two sets of formulas and let $\mathscr{F}$ be a class of frames. We say that Δ *is a consequence of* Γ *w.r.t.* $\mathscr{F}$, and we write $\Gamma \models_{\mathscr{F}} \Delta$, iff, for all models $\underline{K} = \langle P, \leq, \Vdash \rangle$ based on the frames of $\mathscr{F}$ and all $\alpha \in P$, it holds that:

$$\alpha \Vdash A \text{ for all } A \in \Gamma \implies \alpha \Vdash B \text{ for some } B \in \Delta.$$

1.3 Intermediate logics

We denote with **Int** and **Cl** the *propositional intuitionistic logic* and the *propositional classical logic* respectively. An *intermediate propositional logic* L in the language $\mathscr{L}_{\mathscr{V}}$ is any set L of formulas of the language $\mathscr{L}_{\mathscr{V}}$ satisfying the conditions:

- $\mathbf{Int} \subseteq L \subseteq \mathbf{Cl}$;
- L is closed under *modus ponens*;
- L is closed under *substitutions* (i.e., $A \in L$ implies $\sigma A \in L$, for every substitution σ).

Given a set V of propositional variables (contained in the set $\mathscr{V}$), L^V denotes the set of V-formulas of L (note that L^V satisfies all the properties of an intermediate logic with respect to the restricted language $\mathscr{L}_V$).
Given two sets of formulas Γ and Δ, with $\Gamma \vdash_L \Delta$ we mean that there are some formulas $A_1, \ldots, A_n$ in Γ and $B_1, \ldots, B_m$ in Δ such that $A_1 \wedge \cdots \wedge A_n \rightarrow B_1 \vee \cdots \vee B_m \in L$; $\vdash_L A$ means $A \in L$.
In the sequel, we will adopt essentially two ways to define intermediate logics. Let Δ be any set of formulas such that $\Delta \subseteq \mathbf{Cl}$; then $\mathbf{Int} + \Delta$ denotes the intermediate logic L which coincides with the closure of the set of formulas $\mathbf{Int} \cup \Delta$ with respect

to modus ponens and substitutions. The formulas in Δ are called *additional* or *extra axioms* of L (*over* **Int**). If $\Delta = \{A_1, \ldots, A_n\}$, we write also $\mathbf{Int} + A_1 + \cdots + A_n$ instead of $\mathbf{Int} + \Delta$. If a logic L can be represented as $\mathbf{Int} + \Delta$ with Δ finite, we say that L is *finitely axiomatizable*. Given any two intermediate logics L_1 and L_2, $L_1 + L_2$ denotes the union of L_1 with L_2, which is the smallest intermediate logic including both L_1 and L_2.

From a semantical viewpoint, we can define an intermediate logic starting from a nonempty class of frames $\mathscr{F}$. As a matter of fact, let us consider the set:

$$\mathscr{L}(\mathscr{F}) = \{A : \text{for all } \underline{P} = \langle P, \leq \rangle \in \mathscr{F},\ \underline{P} \models A\}.$$

Then, it is well known that $\mathscr{L}(\mathscr{F})$ is an intermediate propositional logic; we call it the *logic of* $\mathscr{F}$.

A logic L is said to be *characterized* (or *described*) by a class of frames $\mathscr{F}$ if it holds that $L = \mathscr{L}(\mathscr{F})$. If $\underline{P} = \langle P, \leq \rangle$ is a frame for a logic L, then any proper generated subframe of $\underline{P}$ is a frame for L; we show that, in some cases, also the converse holds. Let $\alpha \in P$ and let $\underline{P}_\alpha$ be the cone of $\underline{P}$ generated by α; we say that $\underline{P}_\alpha$ has the *filter property* iff, for every $\beta, \gamma \in P$, it holds that:

$$\alpha < \beta \wedge \alpha < \gamma \implies \exists \delta (\alpha < \delta \wedge \delta < \beta \wedge \delta < \gamma).$$

If α is the root of $\underline{P}$, we say that $\underline{P}$ has the filter property.

1.3.1 Proposition *Let L be an intermediate logic, let $\underline{P} = \langle P, \leq, \rho \rangle$ be a frame which has the filter property and suppose that every proper cone of $\underline{P}$ is a frame for L. Then $\underline{P}$ is a frame for L.*

Proof: Suppose, by absurd, that $\underline{P}$ is not a frame for L; then there is a Kripke model $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ based on $\underline{P}$ and a formula $H \in L$ such that $\rho \nVdash H$. We prove the following fact:

(†) There is a model $\underline{K}' = \langle P, \leq, \rho, \Vdash' \rangle$ based on $\underline{P}$ and $\alpha^* \in P$ such that $\rho < \alpha^*$ and $\alpha^* \nVdash' H$.

Let $\mathrm{Sf}(H)$ be the set of all the subformulas of H and let us define a relation $\equiv_H$ between the points δ, δ' of $\underline{P}$ in the following way:

$$\delta \equiv_H \delta' \text{ if and only if, for every } A \in \mathrm{Sf}(H),\ \delta \Vdash A \text{ iff } \delta' \Vdash A.$$

It is immediate to see that $\equiv_H$ is an equivalence relation having finitely many equivalence classes; moreover, if $\alpha > \rho$, then $\alpha \Vdash H$ (indeed, by the hypothesis of the proposition, the cone $\underline{P}_\alpha$ is a frame for L), hence $\alpha \not\equiv_H \rho$. It follows that we can define a set $\mathscr{S} = \{\alpha_1, \ldots, \alpha_n\}$ such that:

(i) $\rho < \alpha_1, \ldots, \rho < \alpha_n$;

(ii) For every $\delta > \rho$ there is $\delta' \in \mathscr{S}$ such that $\delta \equiv_H \delta'$.

Since $\mathscr{S}$ is finite and $\underline{P}$ has the filter property, there exists a point $\alpha^* \in P$ such that $\alpha^* > \rho$ and, for every $\delta \in \mathscr{S}$, $\alpha^* < \delta$. Thus, we can define a model $\underline{K}' = \langle P, \leq, \Vdash' \rangle$ based on $\underline{P}$, where the forcing relation $\Vdash'$ satisfies the following conditions, for every propositional variable p:

- for every $\delta > \alpha^*$, $\delta \Vdash' p$ iff $\delta \Vdash p$;
- $\alpha^* \Vdash' p$ iff $\rho \Vdash p$.

It is immediately verified that:

(A) For every $\delta > \alpha^*$ and every formula A, $\delta \Vdash' A$ iff $\delta \Vdash A$.

From this fact, we can prove, by induction on the complexity of A, that:

(B) For every $A \in \mathrm{Sf}(H)$, $\alpha^* \Vdash' A$ iff $\rho \Vdash A$.

The cases A atomic, $A = B \wedge C$, $A = B \vee C$ are immediate.
Let $A = B \to C$. If $\alpha^* \nVdash' B \to C$, it easily follows, by (A) and by the induction hypothesis, that $\rho \nVdash B \to C$.
Suppose now that $\rho \nVdash B \to C$. Then there is a point δ such that $\alpha \leq \delta$, $\delta \Vdash B$ and $\delta \nVdash C$. If δ coincides with ρ then, being $B, C \in \mathrm{Sf}(H)$, by the induction hypothesis it follows that $\alpha^* \Vdash' B$ and $\alpha^* \nVdash' C$, hence $\alpha^* \nVdash' B \to C$. If $\delta > \rho$, by (ii) there is $1 \leq k \leq n$ such that $\delta \equiv_H \alpha_k$. This implies that $\alpha_k \Vdash B$ and $\alpha_k \nVdash C$, thus, by (i) and (A), $\alpha_k \Vdash' B$ and $\alpha_k \nVdash' C$, hence $\alpha^* \nVdash' B \to C$ and (B) is completely proved.
By (B) and by the fact that $\rho \nVdash H$ and $H \in \mathrm{Sf}(H)$, we get that $\alpha^* \nVdash' H$, thus (†) is proved. By (†), the proper cone $\underline{P}_{\alpha^*}$ of $\underline{P}$ is not a frame for L, in contradiction with the hypothesis of the proposition. We can conclude that $\underline{P}$ is a frame for L. □

1.4 Special sets of formulas

A set of formulas Δ is a L *-saturated set* (in the language $\mathscr{L}_{\mathscr{V}}$) if and only if:

(1) Δ is consistent;

(2) $\Delta \vdash_L A$ (where $A \in \mathscr{L}_{\mathscr{V}}$) implies $A \in \Delta$;

(3) $A \vee B \in \Delta$ implies either $A \in \Delta$ or $B \in \Delta$.

If L is omitted, it is understood that Δ is an **Int**-saturated set. We remark that in (1) *consistent* means that it is not the case that, for some formula A, $\Delta \vdash_L A$ and $\Delta \vdash_L \neg A$; it is well known that such a condition is equivalent to say that $\Delta \nvdash_{CL} A \wedge \neg A$. From (2), it follows that $L \subseteq \Delta$.
The definition of L, V-saturated set is the relativisation of the definition of saturated set with respect to V-formulas; namely, we take into account only the formulas of the language $\mathscr{L}_V$. Note that the set of V-formulas of a L-saturated set Δ is a L, V-saturated set.

We say that Δ is a *maximal consistent set* if Δ is consistent and, for all formulas A, either $A \in \Delta$ or $\neg A \in \Delta$ (which is the same as saying that every $\Delta' \supset \Delta$ is not consistent). One can prove that Δ is a maximal consistent set iff Δ is a **Cl**-saturated set.
We remark that, given a model $\underline{K}$ and a point α of $\underline{K}$, $\Gamma_{\underline{K}}(\alpha)$ is a saturated set and $\Gamma^V_{\underline{K}}(\alpha)$ is a V-saturated set; moreover, if α is a final point of $\underline{K}$, then both $\Gamma_{\underline{K}}(\alpha)$ and $\Gamma^V_{\underline{K}}(\alpha)$ (with respect to V-formulas) are maximal consistent sets. We say that a saturated set Δ is *realized* in $\underline{K}$ if $\Delta = \Gamma_{\underline{K}}(\alpha)$ for some point α of $\underline{K}$ (similar definition for V-saturated sets). We now recall an important lemma about saturated sets (see [4]).

1.4.1 Lemma (Inclusion-exclusion Lemma)
Let L be an intermediate logic and let Γ and Δ be two sets of formulas such that $\Gamma \nvdash_L \Delta$. Then there is a L-saturated set Γ^* such that $\Gamma \subseteq \Gamma^*$ and $\Gamma^* \cap \Delta = \emptyset$. □

1.5 Some notions of completeness

Now we introduce the main notions of the thesis, which refer to the relationships between the syntactical and the semantical aspects of a logic.
Let L be any intermediate propositional logic; a frame $\underline{P} = \langle P, \leq \rangle$ is said to be a *frame for* L if $\underline{P} \models L$; $\mathrm{Fr}(L)$ denotes the class of the frames for L. Note that $\mathrm{Fr}(L)$ is always nonempty, since it contains at least the frame with only one point, which gives rise to classical models.
The following fact immediately follows from the previous definitions.

1.5.1 Proposition *Let L be any intermediate logic, let Γ and Δ be any two sets of formulas. Then:*

(i) $L \subseteq \mathscr{L}(\mathrm{Fr}(L))$.

(ii) $\Gamma \vdash_L \Delta$ *implies* $\Gamma \models_{\mathrm{Fr}(L)} \Delta$.

□

The converse needs not to be true, so the following definitions of *completeness* are justified (see also [15]).

1.5.2 Definition *Let L be any intermediate logic. Then:*

(a) L *is* complete *(or* has Kripke semantics*) iff* $L = \mathscr{L}(\mathrm{Fr}(L))$.

(b) L *is* strongly complete *iff, for any two sets of formulas* Γ *and* Δ*, it holds that:*

$$\Gamma \vdash_L \Delta \Longleftrightarrow \Gamma \models_{\mathrm{Fr}(L)} \Delta.$$

(c) L *is* strongly ω-complete *iff, for every finite set of propositional variables* V*, for any two sets of* V*-formulas* Γ^V *and* Δ^V*, it holds that:*

$$\Gamma^V \vdash_L \Delta^V \Longleftrightarrow \Gamma^V \models_{Fr(L)} \Delta^V.$$

$\square$

In the sequel, we will also refer to these equivalent formulations of the previous definitions.

1.5.3 Proposition *Let* L *be any intermediate logic.*

(i) L *has Kripke semantics if and only if, for every formula* A*, it holds that:*

$$\vdash_L A \iff \models_{\mathrm{Fr}(L)} A.$$

(ii) L *has Kripke semantics if and only if* L *is characterized by some nonempty class of frames.*

(iii) L *is strongly complete if and only if every* L*-saturated set* Δ *is realized in some Kripke model based on a frame for* L*.*

(iv) L *is strongly* ω*-complete if and only if, for every finite* V*, every* L, V*-saturated set* Δ^V *is realized in some Kripke model based on a frame for* L*.*

Proof:
(i) Suppose that $\models_{\mathrm{Fr}(L)} A$ and $\not\vdash_L A$ (recall that $\vdash_L A$ implies $\models_{\mathrm{Fr}(L)} A$). Then $A \in \mathscr{L}(\mathrm{Fr}(L))$ and $A \notin L$, that is $L \neq \mathscr{L}(\mathrm{Fr}(L))$, hence L has not Kripke semantics. Suppose now that L has not Kripke semantics. Then there is a formula A such that $A \in \mathscr{L}(\mathrm{Fr}(L))$ and $A \notin L$, hence $\models_{\mathrm{Fr}(L)} A$ and $\not\vdash_L A$.
(ii) If $L = \mathscr{L}(\mathrm{Fr}(L))$ then L is trivially characterized by the nonempty class of frames $\mathrm{Fr}(L)$. Conversely, let us assume that $L = \mathscr{L}(\mathscr{F})$. Clearly $\mathscr{F} \subseteq \mathrm{Fr}(L)$, which implies that $\mathscr{L}(\mathrm{Fr}(L)) \subseteq \mathscr{L}(\mathscr{F})$, hence $\mathscr{L}(\mathrm{Fr}(L)) = L$.
(iii) Suppose that L is strongly complete and that, by absurd, there is some L-saturated set Δ which is not realized in any model $\underline{K}$ based on a frame for L. Then, if Δ^c is the complement of Δ (i.e. the set of the formulas of $\mathscr{L}_{\mathscr{V}}$ not belonging to Δ), we have:

$$\Delta \models_{\mathrm{Fr}(L)} \Delta^c.$$

Indeed, if $\underline{K}$ is any model based on a frame for L and α is any point of $\underline{K}$ which forces all the formulas of Δ, we must have $\Delta \subset \Gamma_{\underline{K}}(\alpha)$, where the inclusion is proper; this means that $\alpha \Vdash A$ for some $A \notin \Delta$. Since L is strongly complete, it follows that $\Delta \vdash_L \Delta^c$. Hence there are some formulas $A_1, \ldots, A_n$ in Δ and $B_1, \ldots, B_m$ in Δ^c such that $\vdash_L A_1 \wedge \cdots \wedge A_n \rightarrow B_1 \vee \cdots \vee B_m$. This implies that $B_1 \vee \cdots \vee B_m \in \Delta$, thus $B_k \in \Delta$ for some $1 \leq k \leq m$, which is absurd, since $B_k \in \Delta^c$ and $\Delta \cap \Delta^c = \emptyset$. Suppose now that L is not strongly complete. Then there are Γ and Δ such that:

$$\Gamma \models_{\mathrm{Fr}(L)} \Delta \text{ and } \Gamma \nvdash_L \Delta.$$

By the Inclusion-exclusion Lemma, there is a L-saturated set Γ^* such that $\Gamma \subseteq \Gamma^*$ and $\Gamma^* \cap \Delta = \emptyset$. We prove that Γ^* cannot be realized in any model based on a frame for L, and this completes the proof. Suppose, by absurd, that $\Gamma_{\underline{K}}(\alpha) = \Gamma^*$ for some Kripke model $\underline{K}$ and some point α of $\underline{K}$. We get:

- $\alpha \Vdash A$ for every $A \in \Gamma$;
- $\alpha \nVdash B$ for every $B \in \Delta$.

Thus $\Gamma \not\models_{\mathrm{Fr}(L)} \Delta$, in contradiction with the above hypothesis.
(iv) Is proved as (iii). □

1.6 Separability in Kripke models

We introduce some definitions related to the separability of the points of a Kripke model by means of formulas (see also [15]).

1.6.1 Definition *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be any Kripke model and let V be a set of propositional variables.*

(a) $\underline{K}$ *is* (simply) separable *iff, for every $\alpha, \beta \in P$, $\Gamma_{\underline{K}}(\alpha) = \Gamma_{\underline{K}}(\beta)$ implies $\alpha = \beta$.*

(b) $\underline{K}$ *is* (simply) V-separable *iff, for every $\alpha, \beta \in P$, $\Gamma^V_{\underline{K}}(\alpha) = \Gamma^V_{\underline{K}}(\beta)$ implies $\alpha = \beta$.*

(c) $\underline{K}$ *is* well separable *iff, for every $\alpha, \beta \in P$, $\Gamma_{\underline{K}}(\alpha) \subseteq \Gamma_{\underline{K}}(\beta)$ implies $\alpha \leq \beta$.*

(d) $\underline{K}$ *is* well V-separable *iff, for every $\alpha, \beta \in P$, $\Gamma^V_{\underline{K}}(\alpha) \subseteq \Gamma^V_{\underline{K}}(\beta)$ implies $\alpha \leq \beta$.*

(e) $\underline{K}$ *is* full *iff, for every $\alpha \in P$ and every saturated set Δ such that $\Gamma_{\underline{K}}(\alpha) \subseteq \Delta$, there is $\beta \geq \alpha$ such that $\Gamma_{\underline{K}}(\beta) = \Delta$.*

(f) $\underline{K}$ *is* V-full *iff, for every $\alpha \in P$ and every V-saturated set Δ^V such that $\Gamma^V_{\underline{K}}(\alpha) \subseteq \Delta^V$, there is $\beta \geq \alpha$ such that $\Gamma^V_{\underline{K}}(\beta) = \Delta^V$.*

□

We note that in literature (for instance in [4]) separable models are also called *differentiated* or *distinguishable*, well separability is called *tightness*, and full separable models correspond to *descriptive general frames*. The following properties can be easily proved.

1.6.2 Proposition *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a Kripke model and let V be a set of propositional variables.*

(i) If $\underline{K}$ is separable and full, then $\underline{K}$ is well separable and has enough final points.

(ii) If $\underline{K}$ is V-separable and V-full, then $\underline{K}$ is well V-separable and has enough final points.

(iii) If $\underline{K}$ is V-separable and V is finite, then $\underline{K}$ has finitely many final points.

□

A remarkable feature of V-separable and V-full models is that of having maximal points with respect to the forcing of V-formulas in the following sense.

1.6.3 Proposition *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a V-separable and V-full Kripke model, where V is any set of propositional variables; let A be a V-formula and let $\alpha \in P$ be such that $\alpha \nVdash A$. Then there is $\alpha_{MAX} \geq \alpha$ such that $\alpha_{MAX} \nVdash A$ and, for every $\delta > \alpha_{MAX}$, $\delta \Vdash A$.*

Proof: Let us consider the nonempty set

$$\mathscr{D} = \{\Delta^V : \Delta^V \text{ is a } V\text{-saturated set and } \Gamma^V_{\underline{K}}(\alpha) \subseteq \Delta^V \text{ and } A \notin \Delta^V\}.$$

By Zorn Lemma (see for instance [21]) applied to $\mathscr{D}$ (with respect to the partial order $\subseteq$), $\mathscr{D}$ has a maximal element Δ^*. By the V-fullness of $\underline{K}$, there is $\alpha_{MAX} \geq \alpha$ such that $\Gamma^V_{\underline{K}}(\alpha_{MAX}) = \Delta^*$. Suppose that there is $\delta > \alpha_{MAX}$ such that $\delta \nVdash A$. Then $A \notin \Gamma^V_{\underline{K}}(\delta)$, hence $\Gamma^V_{\underline{K}}(\delta) \in \mathscr{D}$; on the other hand, by the V-separability of $\underline{K}$, Δ^* must be properly contained in $\Gamma^V_{\underline{K}}(\delta)$, in contradiction with the fact that Δ^* is maximal in $\mathscr{D}$. Thus the proposition is proved. □

In particular, if V is the set of all the propositional variables, we get:

1.6.4 Proposition *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a separable and full Kripke model, let A be a formula and let $\alpha \in P$ be such that $\alpha \nVdash A$. Then there is $\alpha_{MAX} \geq \alpha$ such that $\alpha_{MAX} \nVdash A$ and, for every $\delta > \alpha_{MAX}$, $\delta \Vdash A$.* □

1.7 The notions of Canonicity and ω-canonicity

We introduce the well known notion of canonicity and its relativized counterpart (see also [15]).

1.7.1 Definition

(A) A logic L is said to be canonical *if and only if every separable and full model of L is based on a frame for L.*

(B) A logic L is said to be ω-canonical *if and only if, for every finite V, every V-separable and V-full model of L^V is based on a frame for L.*

□

The following facts are immediate consequences of the corresponding definitions.

1.7.2 Proposition *Let L be any intermediate logic. Then:*

(i) If L is canonical, then L is ω-canonical.

(ii) If L is strongly complete, then L is strongly ω-complete.

(iii) If L is canonical, then L is strongly complete.

(iv) If L is ω-canonical, then L is strongly ω-complete.

□

The converses of (i) and (ii) do not hold as we will see later. We do not know whether the converses of (iii) and (iv) hold, and this seems to be a difficult open problem.
We recall an important theorem, due to Fine and van-Benthem, which shows how canonicity is closely related to *first-order* definable properties of the classes of frames (for an explanatory discussion about definable first-order properties, see [5]).

1.7.3 Theorem (Van-Benthem) *If a logic L is characterized by a first-order definable class of frames, then L is canonical.* □

Note that, in particular, L has Kripke semantics; in other words, to apply the previous theorem it is not sufficient to single out some first-order definable class $\mathscr{F}$ of frames for L, but it is required to prove, by means of a completeness theorem, that $L = \mathscr{L}(\mathscr{F})$.
As usual, the *canonical model* $\underline{\mathscr{C}}_L = \langle P_L, \leq, \Vdash \rangle$ of a logic L is the Kripke model such that:

- P_L is the set of all the L-saturated sets.
- $\leq$ coincides with the inclusion between sets.
- For every propositional variable p and every $\Delta \in P_L$, $\Delta \Vdash p$ iff $p \in \Delta$.

Using the Inclusion-exclusion Lemma, one can prove that:

- For every formula A and every $\Delta \in P_L$, $\Delta \Vdash A$ iff $A \in \Delta$.

The definition of *V-canonical model* $\mathscr{C}_L^V$ of L (where V is some finite set of propositional variables) is like the definition of canonical model, taking into account the L, V-saturated sets. Clearly, $\mathscr{C}_L$ is a full model of L, even better, it contains (up to isomorphisms, see next section), as generated submodels, all the full models of L; a similar property holds for $\mathscr{C}_L^V$. Thus, denoting with $\underline{P}_L$ and $\underline{P}_L^V$ the frames of $\mathscr{C}_L$ and $\mathscr{C}_L^V$ respectively, we have:

- L is canonical iff $\underline{P}_L$ is a frame for L;
- L is ω-canonical iff, for every finite V, $\underline{P}_L^V$ is a frame for L.

1.8 Morphisms between frames

We present some kinds of morphisms between frames and between models which preserve the validity of formulas. We start by introducing the isomorphisms between frames and models. Two frames $\underline{P} = \langle P, \leq \rangle$ and $\underline{P}' = \langle P', \leq' \rangle$ are said to be *isomorphic* if there is a bijective map f from P onto P' such that, for all $\alpha, \beta, \in P$, it holds that:

$$\alpha \leq \beta \Longleftrightarrow f(\alpha) \leq' f(\beta)$$

In other words, f is an isomorphism between the posets $\underline{P}$ and $\underline{P}'$. Two models $\underline{K} = \langle P, \leq, \Vdash \rangle$ and $\underline{K}' = \langle P', \leq', \Vdash' \rangle$ are said to be isomorphic if there is an isomorphism f between $\underline{P}$ and $\underline{P}'$ such that, for all the propositional variables p of the language and all $\alpha \in P$, it holds that:

$$\alpha \Vdash p \Longleftrightarrow f(\alpha) \Vdash p.$$

This implies that α and $f(\alpha)$ force exactly the same formulas. We point out that isomorphic frames and isomorphic models can be considered indistinguishable.

On the other hand, if the aim is to preserve the validity of formulas, it may be used less powerful maps, such as p-morphisms. Let $\underline{P} = \langle P, \leq \rangle$ and $\underline{P}' = \langle P', \leq' \rangle$ be any two frames; a *p-morphism* from $\underline{P}$ onto $\underline{P}'$ is a surjective map $f : P \to P'$ such that:

(1) f is order preserving;

(2) f is *open*, that is, for every $\alpha \in P$ and $\beta' \in P'$, if $f(\alpha) \leq' \beta'$, then there is $\beta \in P$ such that $\alpha \leq \beta$ and $f(\beta) = \beta'$.

Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ and $\underline{K}' = \langle P', \leq', \Vdash' \rangle$ be any two Kripke models and let V be a set of propositional variables. We say that f is a V *p-morphism* from $\underline{K}$ onto $\underline{K}'$ iff:

(1) f is a p-morphism from $\underline{P}$ onto $\underline{P}'$;

(2) For every $p \in V$ and $\alpha \in P$, $\alpha \Vdash p$ iff $h(\alpha) \Vdash' p$.

From the definition of V p-morphism, we can deduce the following proposition about the conservation of formulas.

1.8.1 Proposition *Let f be a V p-morphism from $\underline{K}$ onto $\underline{K}'$, where V is any set of propositional variables. Then, for every point α of $\underline{K}$, $\Gamma^V_{\underline{K}}(\alpha) = \Gamma^V_{\underline{K}'}(f(\alpha))$.* □

From this proposition, it follows that:

1.8.2 Proposition *Let $\underline{P}$ be a frame and suppose that there is a p-morphism from $\underline{P}$ onto some frame $\underline{P}'$. Then, for every formula A, $\underline{P} \models A$ implies $\underline{P}' \models A$.* □

In particular, if $\underline{P}$ is a frame for a logic L, then also $\underline{P}'$ is a frame for L.
For a frame $\underline{P}$, $Spl(\underline{P})$ denotes the class of frames $\underline{P}'$ such that, for every generated subframe $\underline{P}''$ contained in some cone $\underline{P}'_\alpha$ of $\underline{P}'$, there are no p-morphisms from $\underline{P}''$ onto $\underline{P}$. We can generalize the previous proposition as follows:

1.8.3 Proposition *Let $\underline{P}$ be a frame and let $\underline{P}'$ be a frame such that $\underline{P}' \notin Spl(\underline{P})$. Then, for every formula A, $\underline{P}' \models A$ implies $\underline{P} \models A$.* □

Thus, if $\underline{P}'$ is a frame for a logic L, also $\underline{P}$ is a frame for L.

We recall that, given a model $\underline{K} = \langle P, \leq, \Vdash \rangle$, a standard method to obtain a well V-separable model, in which all the V-saturated sets realized in $\underline{K}$ are represented, is the quotientation of $\underline{K}$ with respect to V-formulas. More formally, let V be any set of propositional variables and let us define the following relation between the points of $\underline{P}$:

$$\alpha \equiv^V \beta \Longleftrightarrow \Gamma^V_{\underline{K}}(\alpha) = \Gamma^V_{\underline{K}}(\beta).$$

It is easy to see that $\equiv^V$ is an equivalence relation. Let us denote with α_V the equivalence class to which the point α of $\underline{P}$ belongs; then, the *quotient model* $\underline{K}_V = \langle P', \leq', \Vdash' \rangle$ of $\underline{K}$ with respect to V is defined as follows:

- $P' = \{\alpha_V : \alpha \in P\}$;
- $\alpha_V \leq' \beta_V$ iff $\Gamma^V_{\underline{K}}(\alpha) \subseteq \Gamma^V_{\underline{K}}(\beta)$;
- For every $p \in V$, $\alpha_V \Vdash' p$ iff $\alpha \Vdash p$;
- For every $p \notin V$, $\alpha_V \nVdash' p$.

One can easily check that the definition is sound; moreover, $\Gamma^V_{\underline{K}}(\alpha) = \Gamma^V_{\underline{K}_V}(\alpha_V)$ and $\underline{K}_V$ is well V-separable. We remark that the map h which associates each α of $\underline{K}$ with the point α_V of $\underline{K}_V$ is surjective, order preserving, but needs not to be open (thus, in general, it is not a p-morphism).

1.9 Finite Kripke models

Finite Kripke models, that is Kripke models with finitely many points, have some remarkable properties. We begin with proving that finite V-separable models are both well V-separable and V-full.

1.9.1 Proposition *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a finite V-separable model, where V is any set of propositional variables. Then $\underline{K}$ is well V-separable.*

Proof: We have to prove that, for every $\alpha \in P$, the following property holds:

(*) for every $\beta \in P$, $\Gamma^V_{\underline{K}}(\alpha) \subseteq \Gamma^V_{\underline{K}}(\beta)$ implies $\alpha \leq \beta$.

We prove (*) by induction on depth(α). Let $\beta \in P$ be such that $\Gamma^V_{\underline{K}}(\alpha) \subseteq \Gamma^V_{\underline{K}}(\beta)$. If depth($\alpha$) $= 1$, that is α is a final point, necessarily $\Gamma^V_{\underline{K}}(\alpha) = \Gamma^V_{\underline{K}}(\beta)$ (in fact, $\Gamma^V_{\underline{K}}(\alpha)$ is maximal between the consistent sets of V-formulas); by the V-separability of $\underline{K}$, $\alpha = \beta$. Suppose that depth(α) > 1 and, by absurd, it is not true that $\alpha \leq \beta$; let $\delta_1, \ldots, \delta_n$ be all the immediate successors of α. Then $\delta_1 \not\leq \beta, \ldots, \delta_n \not\leq \beta$, hence, by the induction hypothesis applied to such points, there are some V-formulas $A_1, \ldots, A_n$ such that $\delta_1 \Vdash A_1, \ldots, \delta_n \Vdash A_n$ and $\beta \nVdash A_1, \ldots, \beta \nVdash A_n$. On the other hand, by the V-separability of $\underline{K}$, there must be a V-formula B such that $\beta \Vdash B$ and $\alpha \nVdash B$. Let C be the V-formula $B \to A_1 \vee \cdots \vee A_n$; we can assert that $\alpha \Vdash C$ and $\beta \nVdash C$, in contradiction with the above assumption. Thus $\alpha \leq \beta$ and (*) is proved. □

1.9.2 Lemma *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a finite model, let V be a set of propositional variables, and let $\underline{K}' = \langle P', \leq', \Vdash' \rangle$ be a Kripke model (possibly $\underline{K}' = \underline{K}$); let $\alpha \in P$ and $\alpha' \in P'$ be such that $\Gamma^V_{\underline{K}}(\alpha) = \Gamma^V_{\underline{K}'}(\alpha')$, and let Δ^V be a V-saturated set such that $\Gamma^V_{\underline{K}'}(\alpha') \subseteq \Delta^V$. Then there is $\beta' \in P'$ such that $\alpha' \leq' \beta'$ and $\Gamma^V_{\underline{K}'}(\beta') = \Delta^V$.*

Proof: By the finiteness of $\underline{K}$, we can find a finite set of V-formulas Σ such that:

- for every V-formula H, there is $A \in \Sigma$ such that $\alpha \Vdash A \leftrightarrow H$.

Let us assume that $\Sigma = \{A_1, \ldots, A_m, B_1, \ldots, B_l\}$, where $A_1 \in \Delta^V, \ldots, A_m \in \Delta^V, B_1 \notin \Delta^V, \ldots, B_l \notin \Delta^V$. By definition of V-saturated set, it follows that the V-formula $Z = A_1 \wedge \cdots \wedge A_m \to B_1 \vee \cdots \vee B_l$ does not belong to Δ^V, hence $\alpha' \nVdash' Z$. This implies that there is $\beta' \in P'$ such that $\alpha' \leq' \beta'$, $\beta' \Vdash' A_1, \ldots, \beta' \Vdash' A_m$ and $\beta' \nVdash' B_1, \ldots, \beta' \nVdash' B_l$. We show that $\Gamma^V_{\underline{K}'}(\beta') = \Delta^V$. Let $H \in \Delta^V$; then, for some $1 \leq i \leq m$, $\alpha \Vdash H \leftrightarrow A_i$, hence $\alpha' \Vdash' H \leftrightarrow A_i$. Since $\beta' \Vdash' A_i$, it follows that $\beta' \Vdash' H$. Likewise we can show that $\Gamma^V_{\underline{K}'}(\beta') \subseteq \Delta^V$; thus $\Gamma^V_{\underline{K}'}(\beta') = \Delta^V$, and the proposition is proved. □

Taking $\underline{K} = \underline{K}'$, it is immediately proved that:

1.9.3 Proposition *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a finite model and let V be any set of propositional variables. Then $\underline{K}$ is V-full.* □

In particular, taking as V the set of all the propositional variables, we get:

1.9.4 Proposition *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a finite Kripke model. Then $\underline{K}$ is full.* □

We now prove the following proposition about p-morphisms onto finite models.

1.9.5 Proposition *Let $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ be a finite V-separable model (where V is any set of propositional variables) and let $\underline{K}' = \langle P', \leq', \rho', \Vdash' \rangle$ be any Kripke model* *such that* $\Gamma^V_{\underline{K}}(\rho) = \Gamma^V_{\underline{K}'}(\rho')$. *Let* h *be a map from the points of* $\underline{K}'$ *to the points of* $\underline{K}$ *defined as follows:*

$$h(\alpha') = \alpha \quad \text{iff} \quad \Gamma^V_{\underline{K}'}(\alpha') = \Gamma^V_{\underline{K}}(\alpha).$$

Then h *is a* V *p-morphism from* $\underline{K}'$ *onto* $\underline{K}$.

Proof: By Lemma 1.9.2 and by the fact that $\underline{K}$ is V-separable, one can easily check that h associates, with each $\alpha' \in P'$, one and only one $\alpha \in P$, h is order preserving and h is open. Thus h is a V p-morphism. □

We now prove that the finite models of a logic L are based on frames for L.

1.9.6 Proposition *Let* L *be an intermediate logic, let* V *be a set of propositional variables and let* $\underline{K} = \langle P, \leq, \Vdash \rangle$ *be a finite* V*-separable model of* L^V. *Then* $\underline{K}$ *is based on a frame for* L.

Proof: Suppose, by absurd, that $\underline{P} = \langle P, \leq \rangle$ is not a frame for L; then there exist a model $\underline{K}' = \langle P, \leq, \Vdash' \rangle$, based on the same frame $\underline{P}$, $\alpha \in P$ and $A \in L$ such that $\alpha \nVdash' A$. Let $V_A = \{q_1, \ldots, q_n\}$ be the set of all the propositional variables occurring in A; by the finiteness and the V-separability of $\underline{K}$, which also imply the well V-separability of $\underline{K}$, we can find some V-formulas $H_1, \ldots, H_n$ which simulate in $\underline{K}$ the forcing of $q_1, \ldots, q_n$ in $\underline{K}'$ respectively; more precisely, for every $\alpha \in P$ and every $1 \leq k \leq n$, it holds that:

$$\alpha \Vdash' q_k \Longleftrightarrow \alpha \Vdash H_k.$$

Let us take any substitution σ such that $\sigma q_k = H_k$ for $1 \leq k \leq n$. Then, for every V_A-formula B and every $\alpha \in P$, by induction on the structure of B, we can prove that:

$$\alpha \Vdash' B \Longleftrightarrow \alpha \Vdash \sigma B.$$

In particular, we get that $\alpha \nVdash \sigma A$. This gives rise to a contradiction; indeed, $A \in L$ implies that $\sigma A \in L$, that is (being σA a V-formula) $\sigma A \in L^V$, thus σA should be valid in $\underline{K}$. □

Taking as V the set of all the propositional variables, we get:

1.9.7 Proposition *Let* L *be an intermediate logic and let* $\underline{K} = \langle P, \leq, \Vdash \rangle$ *be a finite separable model of* L. *Then* $\underline{K}$ *is based on a frame for* L. □

We recall another well known property connected with the finiteness of models.

1.9.8 Proposition *Let* $\underline{K} = \langle P, \leq, \Vdash \rangle$ *be a* V*-separable model, where* V *is a finite set of propositional variables, and let* α *be any point of* $\underline{P} = \langle P, \leq \rangle$ *having finite depth. Then the cone* $\underline{P}_\alpha$ *of* $\underline{P}$ *is finite*

Proof: By induction on depth(α). If depth(α) = 1, that is α is final, the proposition is immediate. Suppose that depth(α) > 1 and let β and γ be two distinct points of $\underline{P}_\alpha$. Since β and γ have different V-forcing, one of the following statements (a) or (b) must hold:

(a) β and γ have different atomic V-forcing;

(b) β and γ do not see the same points.

By the finiteness of V and by the induction hypothesis, only finitely many distinct points can be in $\underline{P}_\alpha$, thus the proposition is proved. □

1.10 Frames with enough final points

We show that, as far as completeness matters are concerned, we can limit ourselves to consider only frames with enough final points (see also [8]). Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a Kripke model and let V be a finite set of propositional variables. We say that a point α of $\underline{K}$ is *V-final* if, for every $p \in V$, either $\alpha \Vdash p$ or $\alpha \Vdash \neg p$. Since V is finite, it is immediate to prove that, for every $\alpha \in P$, there is $\alpha' \geq \alpha$ such that α' is V-final. The *V-pruned* model of $\underline{K}$ is the model $\underline{K}^V_{pr} = \langle P', \leq', \Vdash' \rangle$ obtained by taking the points of $\underline{K}$ up to the V-final points and quotienting the V-final points with respect to the V-formulas. More precisely, for $\alpha \in P$, let us denote with α_V the equivalence class containing the points which have the same V-forcing of α; then $\underline{K}^V_{pr}$ is defined as follows.

(1) $P' = \{\alpha : \alpha \in P \text{ and } \alpha \text{ is not } V\text{-final}\} \cup \{\alpha_V : \alpha \in P \text{ and } \alpha \text{ is } V\text{-final}\}$.

(2) For every $\alpha, \beta \in P$:

- if β is not V-final, then $\alpha \leq' \beta$ iff $\alpha \leq \beta$;
- if β is V-final, then $\alpha \leq' \beta_V$ iff $\Gamma^V_{\underline{K}}(\alpha) \subseteq \Gamma^V_{\underline{K}}(\beta)$ (where the equality holds iff α is V-final and α is equivalent to β).

(3) For every $p \in V$ and $\alpha \in P$:

- if α is not V-final, $\alpha \Vdash' p$ iff $\alpha \Vdash p$;
- if α is V-final, $\alpha_V \Vdash' p$ iff $\alpha \Vdash p$.

(4) For every $p \notin V$ and $\alpha' \in P'$, $\alpha' \Vdash' p$.

It should be clear that $\underline{K}^V_{pr}$ is actually a Kripke model and $\underline{K}^V_{pr}$ has enough final points.

1.10.1 Lemma *Let $\underline{K}$ be a Kripke model and let V be a finite set of propositional variables. Then there is a V p-morphism from $\underline{K}$ onto the V-pruned model $\underline{K}^V_{pr}$ of $\underline{K}$.*

Proof: Let us consider the function f which maps every non V-final point of $\underline{K}$ in itself and every V-final point α in α_V. Then one can easily check that f is a V p-morphism from $\underline{K}$ onto $\underline{K}^V_{pr}$. □

Now we state the main result of this section.

1.10.2 Theorem *Let $\mathscr{F}$ be any class of frames. Then there is a class $\mathscr{F}^{Fin}$, containing only frames with enough final points, such that $\mathscr{L}(\mathscr{F}) = \mathscr{L}(\mathscr{F}^{Fin})$.*

Proof: Let $\mathscr{K}$ be the class of all Kripke models based on the frames of $\mathscr{F}$; let $\mathscr{K}^{Fin}$ be the class containing the V-pruned models of the models of $\mathscr{K}$, for every finite set V; finally, let us take, as $\mathscr{F}^{Fin}$, the class of the frames of the models of $\mathscr{K}^{Fin}$. Suppose that $A \notin \mathscr{L}(\mathscr{F})$ and let $\underline{K} \in \mathscr{K}$ be such that A is not valid in $\underline{K}$. Let $V = \mathrm{Var}(A)$; then, by Lemma 1.10.1, A is not valid in $\underline{K}^V_{pr}$ and, since this model belongs to $\mathscr{K}^{Fin}$, we get that $A \notin \mathscr{L}(\mathscr{F}^{Fin})$. Suppose now that $A \notin \mathscr{L}(\mathscr{F}^{Fin})$; then $\underline{P}^{Fin} \not\models A$, for some $\underline{P}^{Fin} \in \mathscr{F}^{Fin}$. By definition of $\mathscr{F}^{Fin}$, there is $\underline{P} \in \mathscr{F}$ and a p-morphism f from $\underline{P}$ onto $\underline{P}^{Fin}$; this implies that $\underline{P} \not\models A$, hence $A \notin \mathscr{L}(\mathscr{F})$, and this concludes the proof. □

As a consequence of this theorem, we can state that:

- A logic L has Kripke semantics iff L is characterized by some class of frames with enough final points.

Thus, we can assume that a class of frames $\mathscr{F}$ which characterizes a logic L contains only frames with enough final points. Note that the standard tools used in proving the completeness of a logic, by means of canonical (or ω-canonical) models or by filtration techniques, actually refer to classes of frames with enough final points.

Chapter 2

Intermediate Logics

In this chapter we introduce some intermediate logics and we report some facts that will be used later. Thorough this section we essentially follow [4].

2.1 The logics of bounded depth

Let us consider the sequence of formulas $\boldsymbol{bd}_n$ defined as follows:

$$\boldsymbol{bd}_1 = p_1 \vee \neg p_1$$

$$\boldsymbol{bd}_{n+1} = p_{n+1} \vee (p_{n+1} \to \boldsymbol{bd}_n)$$

The family of logics $\mathbf{Bd}_n$ of *bounded depth*, for $n \geq 1$, is defined as follows:

$$\mathbf{Bd}_n = \mathbf{Int} + \boldsymbol{bd}_n.$$

The frames for $\mathbf{Bd}_n$ are the frames of depth at most n, as asserted in next proposition.

2.1.1 Proposition *$\underline{P}$ is a frame for the logic $\mathbf{Bd}_n$, for $n \geq 1$, iff every point of $\underline{P}$ has depth at most n.* □

2.2 The logics of bounded branching

Let us consider the following family of formulas:

$$\boldsymbol{bb}_n = \bigwedge_{i=0}^{n} ((p_i \to \bigvee_{j \neq i} p_j) \to \bigvee_{j \neq i} p_j) \to \bigvee_{i=0}^{n} p_i \qquad n \geq 1$$

The logics $\mathbf{T}_n$, for $n \geq 1$ (also known as *Gabbay - de Jongh logics*, see [13]), are defined as follows:

$$\mathbf{T}_n = \mathbf{Int} + \boldsymbol{bb}_n.$$

Let $\underline{P} = \langle P, \leq \rangle$ be a finite frame and let $\alpha \in P$. We say that α has *branching* n if α has at most n distinct immediate successors. It is not difficult to prove that:

2.2.1 Proposition *Let $\underline{P}$ be a finite frame. $\underline{P}$ is a frame for the logic $\mathbf{T}_n$, with $n \geq 1$, iff every point α of $\underline{P}$ has branching at most n.* □

One can also prove that $\mathbf{T}_n$ is characterized by the class $\mathscr{T}_n$ of n-ary trees (namely, the class of trees whose non-final nodes have exactly n successors). However, the completeness proof (i.e., the proof of $\mathscr{L}(\mathscr{T}_n) \subseteq \mathbf{T}_n$) is not trivial, and the use of suitable filtration techniques is required. This is due to the fact that $\mathbf{T}_n$ is not ω-canonical, as it will be proved in Section 5.3.

2.3 The Dummett logic

The *Dummett logic* (or *chain logic*) is the logic

$$\mathbf{LC} = \mathbf{Int} + (p \to q) \vee (q \to p).$$

We say that a frame $\underline{P} = \langle P, \leq \rangle$ is *strongly connected* if, for every $\alpha, \beta, \gamma \in P$, the following condition holds:

$$\alpha \leq \beta \ \wedge \ \alpha \leq \gamma \implies \beta \ \leq \gamma \ \vee \ \gamma \leq \beta$$

It is easy to prove that the frames for $\mathbf{LC}$ can be characterized as follows:

2.3.1 Proposition *$\underline{P}$ is a frame for the logic $\mathbf{LC}$ iff $\underline{P}$ is strongly connected.* □

We remark that Dummett logic, for its simple semantics, has been deeply investigated in literature and is well known to any people working in intermediate logics. Moreover, it has also been considered for its interest to the computer science community (see [2]).

2.4 The Kreisel-Putnam logic

The *Kreisel-Putnam logic* $\mathbf{KP}$ is the following logic:

$$\mathbf{KP} = \mathbf{Int} + \boldsymbol{kp}$$

where $\boldsymbol{kp}$ is the axiom scheme:

$$\boldsymbol{kp} = (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r).$$

This logic is well known to people working in intermediate propositional logics (see [11, 12, 18]); moreover, it has been the first counterexample to Łukasiewicz's conjecture of 1952 ([22]), asserting that intuitionistic propositional logic is the greatest consistent propositional system L closed under substitution of propositional variables, modus ponens and disjunction property (DP):

(DP) $A \vee B \in L$ implies either $A \in L$ or $B \in L$.

We recall that the propositional logics that satisfy the disjunction property are also known in literature as *constructive logics*. A well known (and easy verifiable) sufficient condition for (DP) is expressed by the following proposition.

2.4.1 Proposition *Let L be an intermediate logic and suppose that $L = \mathscr{L}(\mathscr{F})$ for some nonempty class of frames $\mathscr{F}$. If, for every $\underline{P}_1$ and $\underline{P}_2$ in $\mathscr{F}$, these is $\underline{P} \in \mathscr{F}$ such that both $\underline{P}_1$ and $\underline{P}_2$ are generated subframes of $\underline{P}$, then L satisfies* (DP). □

Using such a condition, one can prove that also the logics $\mathbf{T}_n$ satisfy (DP). On the other hand it is easy to see that the logics $\mathbf{Bd}_n$ and the logic **LC** cannot satisfy (DP) (one has to find suitable instances of the corresponding axiom schema).

As regards the semantics for **KP**, we can prove that (see also [28]):

2.4.2 Proposition *Let $\underline{P} = \langle P, \leq \rangle$ be a frame with enough final points. $\underline{P}$ is a frame for the logic* **KP** *iff, for every $\alpha, \beta, \gamma \in P$, the following condition holds:*

$$\alpha \leq \beta \wedge \alpha \leq \gamma \implies \exists \delta\, (\alpha \leq \delta \wedge \delta \leq \beta \wedge \delta \leq \gamma \wedge Fin(\delta) = Fin(\beta) \cup Fin(\gamma)).$$

□

We point out that it is also possible to characterize, with a more complex first-order sentence, the whole class of frames for **KP**, that is the class containing also the frames without enough final points (see [4, 11, 12]).

2.5 The logics axiomatized by formulas in one variable

In our research, the logics with extra axioms in one variable have a great importance (see also the Introduction). Here we recall some well known facts about these logics (see for instance [1, 4, 15]). In order to describe the non intuitionistically equivalent formulas in one variable p, we consider the **Int**, $\{p\}$-canonical model $\underline{K}_\omega = \langle P_\omega, \leq, \sigma_\omega, \Vdash \rangle$ defined on the frame $\underline{P}_\omega = \langle P_\omega, \leq, \sigma_\omega \rangle$ of Figure 2.1 (straight lines represent the immediate successor relation) and with forcing relation defined on the variable p as follows:

$$\delta \Vdash p \quad \text{iff} \quad \delta = \sigma_1.$$

Let us consider the following sequence of formulas.

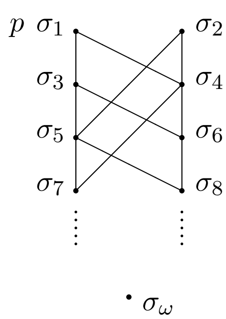

Figure 2.1: The $\{p\}$-canonical model $\underline{K}_\omega$

$$
\begin{aligned}
&\boldsymbol{nf}_1 = p\\
&\boldsymbol{nf}_2 = \neg p\\
&\boldsymbol{nf}_3 = \neg\neg p\\
&\boldsymbol{nf}_4 = \neg\neg p \to p\\
&\boldsymbol{nf}_k = \boldsymbol{nf}_{k-1} \to \boldsymbol{nf}_{k-3} \vee \boldsymbol{nf}_{k-4} \quad \text{for every } k \geq 5.
\end{aligned}
$$

Note that:

$$
\begin{aligned}
&\boldsymbol{nf}_5 = \boldsymbol{nf}_4 \to \boldsymbol{nf}_2 \vee \boldsymbol{nf}_1 = (\neg\neg p \to p) \to \neg p \vee p\\
&\boldsymbol{nf}_6 = \boldsymbol{nf}_5 \to \boldsymbol{nf}_3 \vee \boldsymbol{nf}_2 = ((\neg\neg p \to p) \to \neg p \vee p) \to \neg\neg p \vee \neg p\\
&\boldsymbol{nf}_7 = \boldsymbol{nf}_6 \to \boldsymbol{nf}_4 \vee \boldsymbol{nf}_3 =\\
&\quad (((\neg\neg p \to p) \to \neg p \vee p) \to \neg\neg p \vee \neg p) \to (\neg\neg p \to p) \vee \neg\neg p.
\end{aligned}
$$

The formulas $\boldsymbol{nf}_n$ (possibly with different enumerations) are also known in the literature as *Nishimura-formulas* ([30]). The following facts are well known.

- $\delta \Vdash \boldsymbol{nf}_k$ if and only if $\sigma_k \leq \delta$.
- $\sigma_m \leq \sigma_n$ implies $\vdash_{INT} \boldsymbol{nf}_n \to \boldsymbol{nf}_m$.
- For every $\{p\}$-formula A, there are $n, m \geq 1$ such that $\vdash_{INT} A \leftrightarrow \boldsymbol{nf}_n \vee \boldsymbol{nf}_m$.

Therefore every $\{p\}$-formula is intuitionistically equivalent to some formula of the kind:

$$\boldsymbol{nf}_k\,,\ \boldsymbol{nf}_k \vee \boldsymbol{nf}_{k+1} \quad \text{with } k \geq 1.$$

In correspondence, we can give the following list of superintuitionistic logics in one variable.

- $\mathbf{Int} + (\boldsymbol{nf}_1 \vee \boldsymbol{nf}_2) = \mathbf{Int} + \boldsymbol{nf}_4 = \mathbf{Cl}$.

- $\mathbf{Int} + (\boldsymbol{nf}_2 \vee \boldsymbol{nf}_3) = \mathbf{Int} + \boldsymbol{nf}_5 = \mathbf{Jn}$
  (*Jankov logic* or *Weak excluded middle logic*).
- $\mathbf{NL}_m = \mathbf{Int} + \boldsymbol{nf}_m$, for every $m \geq 6$.
- $\mathbf{NL}_{n,n+1} = \mathbf{Int} + (\boldsymbol{nf}_n \vee \boldsymbol{nf}_{n+1})$, for every $n \geq 3$.

We point out that:

- $\mathbf{NL}_6 = \mathbf{Int} + \boldsymbol{nf}_6 = \mathbf{St}$ is also known as *Scott logic* ([8]);
- $\mathbf{NL}_7 = \mathbf{Int} + \boldsymbol{nf}_7 = \mathbf{Ast}$ is also known as *Anti-Scott logic* ([9]).

All these logics have a simple semantical characterization (see [15]). Let us call $\underline{P}_{\sigma_k}$ the generated subframe of $\underline{P}_\omega$ having root σ_k and $\underline{P}_{\sigma_{k,k+1}}$ the frame obtained by the union of $\underline{P}_{\sigma_k}$ and $\underline{P}_{\sigma_{k+1}}$; let Spl be defined as in Section 1.8. Then:

2.5.1 Proposition *Let $\underline{P}$ be any frame.*

(i) $\underline{P}$ is a frame for the logic $\mathbf{NL}_{m+1}$, for $m \geq 3$, iff $\underline{P} \in Spl(\underline{P}_{\sigma_m})$.

(ii) $\underline{P}$ is a frame for the logic $\mathbf{NL}_{n+1,n+2}$, for $n \geq 1$, iff $\underline{P} \in Spl(\underline{P}_{\sigma_{n,n+1}})$.

□

As a consequence of a result due to Sobolev ([34]), the *finite* frames quoted in the previous proposition characterize the corresponding logics. In next sections, we will show that in some cases we can describe the frames characterizing these logics without any reference to p-morphisms.

2.5.1 The Jankov logic

As seen above, this logic can be axiomatized as follows:

$$\mathbf{Jn} \;=\; \mathbf{Int} + \neg p \vee \neg\neg p \;=\; \mathbf{Int} + \; (\neg\neg p \to p) \to p \vee \neg p.$$

It is trivial to check that these two axiomatizations actually lead to the same logic. In fact, it is immediately verified that:

$$\vdash_{INT} (\neg p \vee \neg\neg p) \;\to\; ((\neg\neg p \to p) \to p \vee \neg p).$$

Moreover, the instance $\neg A \vee \neg\neg A$ of the former axiom scheme can be intuitionistically derived starting from the instance $(\neg\neg\neg A \to \neg A) \to \neg A \vee \neg\neg A$ of the latter one. We also point out that the axiom schema $\neg p \vee \neg\neg p$ is also called the *weak law of excluded middle* (see [17]). We say that a frame $\underline{P} = \langle P, \leq \rangle$ is *strongly directed* if, for every $\alpha, \beta, \gamma \in P$, the following condition holds:

$$\alpha \leq \beta \;\wedge\; \alpha \leq \gamma \;\Longrightarrow\; \exists \delta\, (\beta \leq \delta \wedge \gamma \leq \delta).$$

It is easy to prove that:

2.5.2 Proposition *$\underline{P}$ is a frame for the logic* $\mathbf{Jn}$ *iff $\underline{P}$ is strongly directed.* □

2.5.2 The Scott logic

The Scott Principle, corresponding to the formula $\boldsymbol{nf}_6$ (and to the formula F_9 of [1]), is the extra intuitionistic axiom schema of a well known intermediate constructive propositional logic quoted in the paper [18] as another example contradicting Lukasiewicz's conjecture. This principle has been extensively studied by people working in intermediate propositional logics (see [7, 28, 29]). In [8] (where a Kripke frame semantics formerly introduced in [28] is shown to be valid and complete for **St**) a logic extending **St** and maximal in the family of intermediate constructive propositional logics is exhibited (where a *maximal (propositional intermediate) constructive logic* is a propositional intermediate logic which satisfies (DP) and does not admit any constructive extension). In the framework of intermediate propositional logics, Scott Logic is maximal in the fragment in one variable; moreover, as shown in [9], the fragment in one variable of any propositional constructive logic is either contained in the fragment in one variable of propositional **St**, or it is contained in the fragment in one variable of the propositional intermediate logic **Ast**, the latter being the logic having as extra intuitionistic axiom the formula $\boldsymbol{nf}_7$. The name **Ast** means "anti" Scott: this refers to the fact that the logics **St** and **Ast** are constructively incompatible, that is, the union of these two formal systems gives rise to an intermediate propositional logic which does not admit a constructive extension (in [9] also a maximal intermediate constructive logic is exhibited which includes **Ast**).

Here, we are interested in giving a characterization of the frames for **St** with bounded depth; we show that the characterization of finite frames for **St** given in [8] can be extended to the class of frames for **St** of finite depth. Let $\underline{P} = \langle P, \leq \rangle$ be a frame, let α be a non-final point of $\underline{P}$ and let φ and ψ be two final points of $\underline{P}$. We say that α is *prefinal* iff, for every $\delta > \alpha$, δ is final. We say that φ and ψ are *prefinally connected in* $\underline{P}$ iff either $\varphi = \psi$ or there is a sequence $\varphi_1, \ldots, \varphi_n$ $(n > 1)$ of final points of $\underline{P}$ satisfying the following conditions:

(1) $\varphi_1 = \varphi$ and $\varphi_n = \psi$;

(2) for every $1 \leq i \leq n-1$, there is $\alpha \in P$ such that α is prefinal and $\{\varphi_i, \varphi_{i+1}\} \subseteq \mathrm{Fin}(\alpha)$.

2.5.3 Proposition *Let $\underline{P} = \langle P, \leq \rangle$ be a frame having finite depth. $\underline{P}$ is a frame for the logic* **St** *iff, for every $\alpha \in P$ and for every φ and ψ belonging to Fin(α), φ and ψ are prefinally connected in $\underline{P}_\alpha$.*

Proof: Firstly we observe that, if there are in $\underline{P}$ a non-final point α and a final point $\varphi \in \mathrm{Fin}(\alpha)$ such that, for every $\alpha \leq \beta < \varphi$, β is not prefinal, then $\underline{P}$ is not a frame for **St**. As a matter of fact, since α has finite depth, there is β such that $\alpha \leq \beta$ and φ is an immediate successor of β. Moreover there must be a non-final immediate successor γ of β such that $\varphi \notin \mathrm{Fin}(\gamma)$. It is not difficult to define a p-morphism h from the cone $\underline{P}_\beta$ of $\underline{P}$ onto $\underline{P}_{\sigma_5}$ such that $h(\varphi) = \sigma_2$, $h(\gamma) = \sigma_3$ and $h(\beta) = \sigma_5$, and this means that $\underline{P}$ is not a frame for **St**. Let us assume that there are $\alpha \in P$, φ and

ψ in $\mathrm{Fin}(\alpha)$ such that φ and ψ are not prefinally connected in $\underline{P}_\alpha$; we prove that $\underline{P}$ is not a frame for **St**. By the previous observation, we have only to consider the case where, for every non-final $\delta \geq \alpha$ and every final $\varphi' \in \mathrm{Fin}(\delta)$, there is a prefinal β such that $\delta \leq \beta < \varphi'$. Let Φ be the set of all the final points φ' of $\underline{P}_\alpha$ which are prefinally connected with φ (note that $\varphi \in \Phi$ and $\psi \notin \Phi$). We define a map h from $\underline{P}_\alpha$ onto $\underline{P}_{\sigma_5}$ in the following way.

- For every final point φ' of $\underline{P}_\alpha$:

  $h(\varphi') = \sigma_1$ if $\varphi' \in \Phi$;
  $h(\varphi') = \sigma_2$ otherwise.

- For every non-final point δ of $\underline{P}_\alpha$:

  $h(\delta) = \sigma_3$ if $\mathrm{Fin}(\delta) \subseteq \Phi$;
  $h(\delta) = \sigma_2$ if $\mathrm{Fin}(\delta) \cap \Phi = \emptyset$;
  $h(\delta) = \sigma_5$ otherwise.

Note that $h(\varphi) = \sigma_1$, $h(\psi) = \sigma_2$ and $h(\alpha) = \sigma_5$. It is not difficult to check that, in our hypothesis, h is a p-morphism from $\underline{P}_\alpha$ onto $\underline{P}_{\sigma_5}$, hence $\underline{P}$ is not a frame for **St** and the "only if" part of the proposition is proved.
Conversely, let us assume that $\underline{P}$ is not a frame for **St**; then there are $\alpha \in P$ and a p-morphism from $\underline{P}_\alpha$ onto $\underline{P}_{\sigma_5}$. Let us take two final points φ and ψ such that $h(\varphi) = \sigma_1$ and $h(\psi) = \sigma_2$; it is not difficult to prove that φ and ψ cannot be prefinally connected, and this concludes the proof. □

We remark that the condition of "prefinal connection" cannot be expressed by a first-order formula, and the problem lies in the unbounded number of final points involved in the definition. A formal proof of this fact can be accomplished by a standard application of the classical Compactness theorem (see for instance [5]).

2.5.3 The Anti-Scott logic

The frames for **Ast** with finite depth satisfy a condition which can be expressed by a first-order sentence, as in the statement of next proposition (see also [9], where such a condition is introduced to characterize the finite frames of **Ast**).

2.5.4 Proposition *Let $\underline{P} = \langle P, \leq \rangle$ be a frame having finite depth. $\underline{P}$ is a frame for the logic* **Ast** *if and only if, for every $\alpha \in P$, if α is a non-final point of $\underline{P}$, then one of the following conditions (a) or (b) is satisfied.*

(a) For every immediate successor δ of α, $|Fin(\delta)| = 1$.

(b) For any two immediate successors β and γ of α in $\underline{P}$, if β and γ are non-final, then $Fin(\beta) = Fin(\gamma)$.

Proof: Let us suppose that, for some non-final point α of $\underline{P}$, both (a) and (b) do not hold in $\underline{P}$; we prove that $\underline{P}$ is not a frame for **Ast**. We have that:

(A) There is an immediate successor δ of α which has at least two distinct final points.

(B) There are two non-final immediate successors β and γ of α such that $\mathrm{Fin}(\beta) \neq \mathrm{Fin}(\gamma)$.

Without loss of generality, we can assume that there is a final point φ_1 of $\underline{P}_\alpha$ such that $\varphi_1 \in \mathrm{Fin}(\beta)$ and $\varphi_1 \notin \mathrm{Fin}(\gamma)$. Let us distinguish two cases (C1) and (C2).

(C1) $\mathrm{Fin}(\beta) = \{\varphi_1\}$.

By (A), there is $\varphi_2 \in \mathrm{Fin}(\delta)$ such that $\varphi_2 \neq \varphi_1$. Let us define a map h from $\underline{P}_\alpha$ to $\underline{P}_{\sigma_6}$ in the following way.

- $h(\varphi_2) = \sigma_2$.
- For every final point $\varphi' \neq \varphi_2$, $h(\varphi') = \sigma_1$.
- $h(\beta) = \sigma_3$.
- For every ε such that $\alpha < \varepsilon$ and $\varepsilon \neq \beta$:

  $h(\varepsilon) = \sigma_1$ if $\varphi_2 \notin \mathrm{Fin}(\varepsilon)$;

  $h(\varepsilon) = \sigma_2$ if $\mathrm{Fin}(\varepsilon) = \{\varphi_2\}$;

  $h(\varepsilon) = \sigma_4$ otherwise.
- $h(\alpha) = \sigma_6$.

Note that $h(\varphi_1) = \sigma_1$ and $h(\delta) = \sigma_4$. It is easy to prove that h is a p-morphism from $\underline{P}_\alpha$ onto $\underline{P}_{\sigma_6}$; this means that $\underline{P}$ is not a frame for **Ast**.

(C2) $\mathrm{Fin}(\beta) \neq \{\varphi_1\}$.

In this case $\mathrm{Fin}(\beta)$ contains a point φ different from φ_1. Then we can define a p-morphism from $\underline{P}_\alpha$ onto $\underline{P}_{\sigma_6}$ which maps φ_1 in σ_2, all the other final points in σ_1, β in σ_4, γ alone in σ_3 and α alone in σ_6; also in this case $\underline{P}$ is not a frame for **Ast** and the "only if" part is completely proved.
Conversely, let us suppose that $\underline{P}$ is not a frame for **Ast**; then there is $\alpha \in P$ and a p-morphism h from $\underline{P}_\alpha$ onto $\underline{P}_{\sigma_6}$. Since α has finite depth, we can assume without loss of generality that, for every $\beta > \alpha$, $h(\beta) \neq \sigma_6$; this implies that α is not final and neither (a) nor (b) hold on α. □

2.6 The Medvedev logic

The Medvedev logic **MV** is known in literature as the logic of *finite problem* ([25, 26, 27]), and it arises in the framework of algorithmic interpretation of intuitionistic connectives (see [4] for more references). Here we are interested in the Kripke semantics of such a logic. Let X be nonempty finite set; the *Medvedev frame* (shortly, *MV-frame*) determined by X, is the frame $\underline{P} = \langle P, \leq \rangle$ defined as follows:

- $P = \{Y : Y \subseteq X \quad \text{and} \quad Y \neq \emptyset\}$.

- $Y \leq Z$ iff $Z \subseteq Y$.

Note that X is the root of $\underline{P}$, while the sets $\{x\}$, for each $x \in X$, are the final points of $\underline{P}$. For instance, for $X_2 = \{a, b\}$ and $X_3 = \{a, b, c\}$, the corresponding MV-frames are represented in Figure 2.2. Let $\mathscr{F}_{MV}$ be the class of all the MV-frames; then:

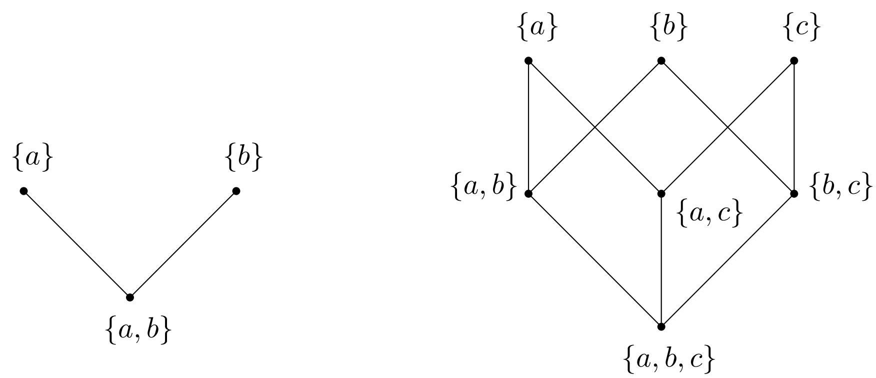

Figure 2.2: The MV-frames with 2 and 3 final points

$\mathbf{MV} = \mathscr{L}(\mathscr{F}_{MV})$.

By this semantical characterization, it immediately follows that:

- $\mathbf{St} \subseteq \mathbf{MV}$;

- $\mathbf{KP} \subseteq \mathbf{MV}$.

By Proposition 2.4.1, it is immediately proved that **MV** satisfy (DP). Moreover, **MV** has been the first example in literature of maximal constructive logic ([20, 23]).

Despite the great interest in literature about this logic, no axiomatization is known; we only know that **MV** is not finitely axiomatizable ([24]); thus the problem of its decidability is still open.

2.7 The logic of rhombuses

The so called *logic of rhombuses* **RH** presents some analogies with Medvedev logic, even if it is less known and less investigated in literature. As for **MV**, we give a semantical characterization (see [3, 4, 9, 23]). Let T be a linear ordering; an *interval* of T is a pair $[t_1, t_2]$, with $t_1 \leq t_2$ (we will denote the interval $[t, t]$ simply with t). The ordering on T induces, in an obvious way, a partial ordering $\subseteq$ on the intervals of T (which intuitively corresponds to the containment relation) defined in the following way:

$$[t_1, t_2] \subseteq [u_1, u_2] \text{ iff } u_1 \leq t_1 \text{ and } t_2 \leq u_2.$$

Let T be a finite linear ordering; the RH-frame $\underline{P} = \langle P, \leq \rangle$ determined by T is defined as follows:

- $P = \{[t_1, t_2] : t_1, t_2 \in T \text{ and } t_1 \leq t_2\}$;
- $[t_1, t_2] \leq [u_1, u_2]$ iff $[u_1, u_2] \subseteq [t_1, t_2]$.

Note that the intervals of the kind $[t, t]$ are the final points of $\underline{P}$ and the interval corresponding to the endpoints of T is the root of $\underline{P}$. For instance, if $T = \{t_1, t_2, t_3, t_4\}$, with $t_1 < t_2 < t_3 < t_4$, the RH-frame determined by T looks as in Figure 2.3. Let

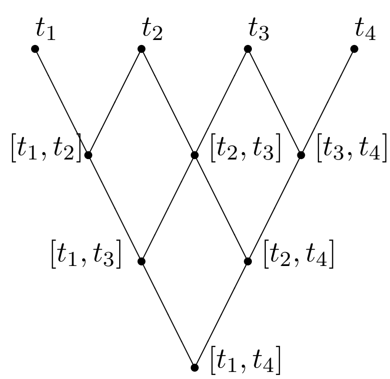

Figure 2.3: The RH-frame with 4 final points

$\mathscr{F}_{RH}$ be the class of all the RH-frames; then:

$$\mathbf{RH} = \mathscr{L}(\mathscr{F}_{RH}).$$

It follows that:

- $\mathbf{St} \subseteq \mathbf{RH}$;
- $\mathbf{T}_2 \subseteq \mathbf{RH}$.

The logics **MV** and **RH** are incomparable; indeed, **KP** is not contained in **RH**, while $\mathbf{T}_2$ is not contained in **MV**. In contrast with the conjecture in [23], **RH** is not a maximal among the logics with (Dp); as a matter of fact, in [9] it is exhibited a logic which has (Dp) and properly extends **RH**. Also for this logic no axiomatization is known.

Chapter 3

Canonicity and ω-Canonicity

In this chapter we will give a more refined classification of Canonicity and ω-Canonicity, according to the lines followed in [15].

3.1 Kinds of canonicity

In order to prove the canonicity of a logic L, not all the properties involved in the corresponding definition are required. According to [15], we propose a finer classification of canonical logics based on the possibility of weakening the requirements to be satisfied by the models $\underline{K}$ without affecting canonicity.

3.1.1 Definition *Let L be any intermediate logic.*

(a) L is quasi hypercanonical of type 0 *($QHYP_0$) iff the underlying frame of any separable Kripke model of L is a frame for L.*

(b) L is quasi hypercanonical of type 1 *($QHYP_1$) iff the underlying frame of any well separable Kripke model of L is a frame for L.*

(c) L is quasi hypercanonical of type 2 *($QHYP_2$) iff the underlying frame of any separable Kripke model of L which has enough final points is a frame for L.*

(d) L is extensively canonical *iff the underlying frame of any Kripke model of L which is well separable and has enough final points is a frame for L.*

□

Mutual relations between these notions are depicted by the diagram in Figure 3.1, in which arrows represent the inclusion relation. We give some insights about these

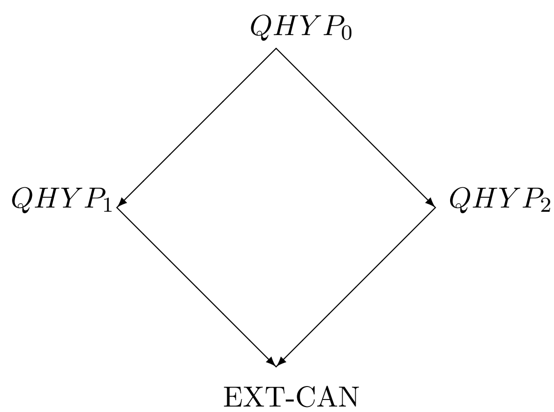

Figure 3.1: Kinds of canonicity

notions by means of a few examples. More precisely, we revise some well known proofs of canonicity putting into evidence the minimal properties required in the proof.
First of all, as an example of logics whose canonicity proof requires weak properties on models, we quote the family of logics of bounded depth $\mathbf{Bd}_h$.

3.1.2 Theorem *The logics* $\mathbf{Bd}_h$, *for every* $h \geq 1$, *are* $QHYP_0$.

Proof: Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a separable model of the logic $\mathbf{Bd}_h$. If $\underline{P}$ is not a frame for $\mathbf{Bd}_h$, we can find $h+1$ distinct points $\alpha_1, \ldots, \alpha_{h+1}$ of $\underline{P}$ such that $\alpha_1 < \cdots < \alpha_{h+1}$. By the separability of $\underline{K}$, we can find h formulas $A_1, \ldots, A_h$ such that, for every $1 \leq m \leq h$, it holds that:

- $\alpha_m \nVdash A_m$;
- $\alpha_{m+1} \Vdash A_m$.

It follows that α_1 does not force the instance of the axiom schema $\boldsymbol{bd}_h$ obtained by replacing p_m with A_m, a contradiction □

As an immediate consequence, we can state the following completeness theorem.

3.1.3 Corollary *The logic* $\mathbf{Bd}_h$, *with* $h \geq 1$, *is characterized by the class of frames with depth at most* h. □

An example of quasi hypercanonical logic of type 1 is the Dummett logic **LC**.

3.1.4 Theorem *The Dummett logic* **LC** *is* $QHYP_1$.

Proof: Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a well separable model of **LC**. If $\underline{P}$ is not a frame for **LC**, then there are $\alpha, \beta, \gamma \in P$ such that $\alpha \leq \beta$, $\alpha \leq \gamma$, $\beta \not\leq \gamma$ and $\gamma \not\leq \beta$. By the well separability of $\underline{K}$, there are two formulas A and B such that:

- $\beta \Vdash A$ and $\gamma \nVdash A$;
- $\gamma \Vdash B$ and $\beta \nVdash B$.

Thus α cannot force the instance $(A \to B) \vee (B \to A)$ of the axiom scheme of **LC**, a contradiction. □

3.1.5 Corollary *The logic* **LC** *is characterized by the class of frames strongly connected.* □

We now exhibit a model of **LC** based on a frame which is not strongly connected. Let $\underline{P} = \langle P, \leq \rangle$ be the frame defined as in the left hand part of Figure 3.2. that is:

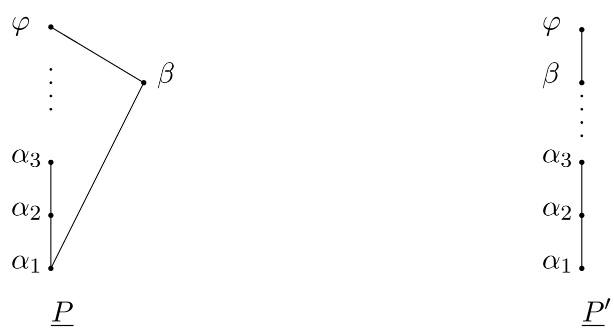

Figure 3.2: The frames $\underline{P}$ and $\underline{P}'$

- $P = \{\alpha_k : k \geq 1\} \cup \{\beta, \varphi\}$;
- $\alpha_1 < \alpha_2 < \ldots < \alpha_k < \ldots < \varphi$;
- $\alpha_1 < \beta < \varphi$ and, for every $k \geq 2$, $\alpha_k \not< \beta$ and $\beta \not< \alpha_k$.

Let us consider the model $\underline{K} = \langle P, \leq, \Vdash \rangle$ based on the frame $\underline{P}$ whose forcing relation $\Vdash$ between the points of $\underline{P}$ and the propositional variables $p_0, p_1, \ldots$ of the language is defined as follows:

- for every $k \geq 1$ and every $\delta \in P$, $\delta \Vdash p_k$ iff either $\delta \geq \alpha_k$ or $\delta = \beta$;
- $\delta \Vdash p_0$ iff $\delta = \varphi$.

It is immediate to see that $\underline{K}$ is separable (in fact, any two distinct points of $\underline{K}$ are separated by a propositional variable) and has enough final points. Let $\underline{P}' = \langle P', \leq' \rangle$ be the linear frame in the right hand part of the figure above (where P' coincides with P) and let $\underline{K}'$ be the model $\underline{P}' = \langle P', \leq', \Vdash' \rangle$, where $\Vdash'$ is defined as $\Vdash$. Let, for each $k \geq 0$, $V_k = \{p_0, \ldots, p_k\}$; then the following properties hold.

(A) For every $k \geq 1$, $\Gamma^{V_k}_{\underline{K}'}(\alpha_k) = \Gamma^{V_k}_{\underline{K}'}(\beta)$.

The proof of (A) is an easy induction on the structure of the formulas. From (A), by induction on the complexity of the formulas, it follows that:

(B) For every $\delta \in P$, $\Gamma_{\underline{K}}(\delta) = \Gamma_{\underline{K}'}(\delta)$.

Let us consider the most interesting case where, for $k \geq 1$, $\alpha_k \nVdash' A \to B$ and we have to prove that $\alpha_k \nVdash A \to B$. Let $\delta \geq' \alpha_k$ be such that $\delta \Vdash' A$ and $\delta \nVdash' B$; if either $\delta = \alpha_m$, for some $m \geq k$, or $\delta = \varphi$, by the induction hypothesis we immediately have that $\delta \Vdash A$ and $\delta \nVdash B$, hence $(\alpha_k \leq \delta)$ $\alpha_k \nVdash A \to B$. It remains to consider the case $\delta = \beta$. By (A), we can find $j \geq k$ such that $\alpha_j \Vdash' A$ and $\alpha_j \nVdash' B$; as before, we can conclude that $\alpha_k \nVdash A \to B$.
By (B), it follows that $\Gamma_{\underline{K}}(\alpha_1) = \Gamma_{\underline{K}'}(\alpha_1)$ and, since $\underline{K}'$ is evidently a model of **LC**, we can state that $\underline{K}$ is a model of **LC** as well. Finally, since $\underline{P}$ is not strongly connected, we can conclude that:

3.1.6 Theorem *The logic* **LC** *is not* $QHYP_2$. □

Note however that $\underline{K}$ is not well separable; for instance, for every $k \geq 1$ $\Gamma_{\underline{K}}(\alpha_k) \subseteq \Gamma_{\underline{K}}(\beta)$, but $\alpha_k \not\leq \beta$.

An example of $QHYP_2$ logic is the Jankov logic **Jn**.

3.1.7 Theorem *The logic* **Jn** *is* $QHYP_2$.

Proof: Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a separable model of **Jn** having enough final points and let us suppose that $\underline{P}$ is not strongly directed. Then there must be $\alpha, \beta, \gamma \in P$ such that:

- $\alpha \leq \beta$ and $\alpha \leq \gamma$;
- for every $\delta \in P$, either $\beta \not\leq \delta$ or $\gamma \not\leq \delta$.

Let, by definition of $\underline{K}$, φ_1 and φ_2 be two final points such that $\beta \leq \varphi_1$ and $\gamma \leq \varphi_2$. By the above assumption, it is not true that $\beta \leq \varphi_2$, hence φ_1 and φ_2 are distinct; by the separability of $\underline{K}$, there exists a formula A such that $\varphi_1 \Vdash A$ and $\varphi_2 \nVdash A$, that is $\varphi_2 \Vdash \neg A$. Thus it is not the case that $\alpha \Vdash \neg A \vee \neg\neg A$, and so an instance of the axiom schema of **Jn** is not valid in $\underline{K}$. □

3.1.8 Corollary *The logic* **Jn** *is characterized by the class of frames strongly directed.* □

We give now an example of a well separable model $\underline{K} = \langle P, \leq, \Vdash \rangle$ of **Jn** based on a frame which is not strongly directed. Let us consider the frame $\underline{P} = \langle P, \leq \rangle$ (see Figure 3.3), where:

- $P = \{\rho\} \cup \{\alpha_k : k \geq 0\} \cup \{\beta_k : k \geq 0\}$;
- $\alpha_k \leq \delta$ iff $\delta = \alpha_m$ and $k \leq m$;
- $\beta_k \leq \delta$ iff $\delta = \beta_m$ and $k \leq m$;
- $\rho \leq \delta$ for every $\delta \in P$.

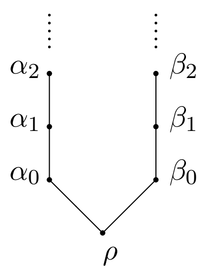

Figure 3.3: The non strongly directed frame $\underline{P}$

Let us divide the variables of the language in the two disjoint countable sets $p_0, p_1, \ldots$ and $q_0, q_1, \ldots$, and let $V_k = \{p_0, \ldots, p_k, q_0, \ldots, q_k\}$. We define the forcing relation $\Vdash$ as follows:

- $\delta \Vdash p_k$ iff either $\alpha_k \leq \delta$ or $\beta_0 \leq \delta$;
- $\delta \Vdash q_k$ iff either $\beta_k \leq \delta$ or $\alpha_0 \leq \delta$.

It is easy to prove that:

$$\Gamma^{V_k}_{\underline{K}}(\alpha_k) = \Gamma^{V_k}_{\underline{K}}(\beta_k).$$

This implies that, for every formula A, $\rho \Vdash \neg A \vee \neg\neg A$, hence $\underline{K}$ is a model of **Jn**. In order to prove that $\underline{K}$ is well separable, it suffices to observe that, for each pair of points α_j and β_l $(j, l \geq 0)$, the following facts hold:

- $\alpha_j \Vdash q_{l+1}$ and $\beta_l \nVdash q_{l+1}$;
- $\beta_l \Vdash p_{j+1}$ and $\alpha_j \nVdash p_{j+1}$.

Therefore, any two points of $\underline{K}$ are well separated by a propositional variable. We can conclude that $\underline{K}$ is a well separable model of **Jn**, whose underlying frame is not strongly directed; then:

3.1.9 Theorem *The logic* **Jn** *is not* $QHYP_1$. □

Note however that $\underline{K}$ has not any final point.

We could continue along these lines and, for instance, provide examples of extensively canonical logics which are neither $QHYP_1$ nor $QHYP_2$; on the other hand, in the sequel, we are not interested in such a refined classification. Indeed, we will take into consideration only one kind of hypercanonicity, according to the following definition:

- Let L be any intermediate logic. L is *hypercanonical* iff the underlying frame of any separable Kripke model of L with enough final points is a frame for L.

Note that hypercanonicity coincides with the definition of $QHYP_2$. On the other hand, since hereafter we will assume that Kripke models have enough final points, hypercanonicity becomes the most elementary kind of canonicity, while the set of the $QHYP_1$ logics collapses into the set of the extensively canonical logics. Moreover, we consider elementary also extensive canonicity, whereas the relevant gap stands in the passage from extensive canonicity to the "simple" canonicity which requires the fullness of models.

An example of logic which is canonical, but not extensively canonical (and where the canonicity proof requires more sophisticated arguments than the ones used so far) is the Kreisel-Putnam logic **KP**. We point out that the canonicity of **KP** is a well-known property. For instance, it can be derived by the fact that the class of frames for **KP** is first-order definable (see Section 2.4) by applying Van-Benthem Theorem (Theorem 1.7.3), or it can be stated as a consequence of a general theorem about canonicity explained in [14]. Here we give a direct proof in order to put into evidence the role played by the fullness property.

3.1.10 Theorem *The logic* **KP** *is canonical.*

Proof: Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a separable and full model of **KP** and let $\alpha, \beta, \gamma \in P$ be such that $\alpha \leq \beta$ and $\alpha \leq \gamma$. We have to show that there is $\delta \in P$ such that:

(a) $\alpha \leq \delta$, $\delta \leq \beta$ and $\delta \leq \gamma$;

(b) $\mathrm{Fin}(\delta) = \mathrm{Fin}(\beta) \cup \mathrm{Fin}(\gamma)$.

Let us take the set:

$$\Delta = \{A : \ \Gamma(\alpha) \cup \mathrm{Neg}_{\beta,\gamma} \ \vdash_{INT} \ A\}$$

where $\mathrm{Neg}_{\beta,\gamma}$ denotes the set of the negated formulas belonging to $\Gamma(\beta) \cap \Gamma(\gamma)$.

(i) Δ is a **KP**-saturated set.

It is immediate to prove that Δ is consistent, Δ is closed under provability and $\mathbf{KP} \subseteq \Delta$. Suppose now that $A \vee B \in \Delta$; then, for some $C_1, \ldots, C_n \in \Gamma(\alpha)$ and $\neg H_1, \ldots, \neg H_m \in \mathrm{Neg}_{\beta,\gamma}$, we have that $C_1 \wedge \cdots \wedge C_n \wedge \neg H_1 \wedge \cdots \wedge \neg H_m \vdash_{INT} A \vee B$. Let H be the formula $\neg H_1 \wedge \cdots \wedge \neg H_m$; since $\vdash_{INT} \neg\neg H \leftrightarrow H$, we get that $\alpha \Vdash \neg\neg H \rightarrow$

$A \vee B$ and, since α forces all the instances of $\boldsymbol{kp}$, $\alpha \Vdash (\neg\neg H \to A) \vee (\neg\neg H \to B)$. Let us suppose, by definiteness, that $\alpha \Vdash \neg\neg H \to A$; then $\neg\neg H \to A \in \Gamma(\alpha)$, which implies that $\Gamma(\alpha) \cup \text{Neg}_{\beta,\gamma} \vdash_{INT} A$, that is $A \in \Delta$. Thus (i) is proved. By definition of Δ, we immediately have that:

(ii) $\Gamma(\alpha) \subseteq \Delta$, $\Delta \subseteq \Gamma(\beta)$ and $\Delta \subseteq \Gamma(\gamma)$.

We now show that:

(iii) For every consistent maximal set Φ, $\Delta \subseteq \Phi$ iff either $\Gamma(\beta) \subseteq \Phi$ or $\Gamma(\gamma) \subseteq \Phi$.

By (ii) the "if" part is immediate. Suppose now that $\Delta \subseteq \Phi$, $\Gamma(\beta) \not\subseteq \Phi$ and $\Gamma(\gamma) \not\subseteq \Phi$. Then there are two formulas H and K such that $H \in \Gamma(\beta)$, $K \in \Gamma(\gamma)$, $H \notin \Phi$ and $K \notin \Phi$. It follows that $\neg\neg(H \vee K) \in \text{Neg}_{\beta,\gamma}$, hence $\neg\neg(H \vee K) \in \Delta$ and $\neg\neg(H \vee K) \in \Phi$. Since Φ is a consistent maximal set (hence a **Cl**-saturated set) it holds that $H \vee K \in \Phi$, hence either $H \in \Phi$ or $K \in \Phi$, in contradiction with the above assumption; thus (iii) is proved.
By (i) and by the fullness of $\underline{K}$, there is $\delta \in P$ such that $\Gamma(\delta) = \Delta$; by (ii), (iii), and by the separability and the fullness of $\underline{K}$, δ satisfies the Conditions (a) and (b), and this completes the proof. □

3.1.11 Corollary *The logic* **KP** *is characterized by the class of frames with enough final points which satisfy the condition of Proposition 2.4.2.* □

In Section 5.3 it will be proved that **KP** is not extensively ω-canonical, and this implies that **KP** is not even extensively canonical.

3.2 Kinds of ω-canonicity

We can now repeat the discussion about canonicity in the context of the ω-canonicity. Clearly some notions collapse, due to the fact that, for V-finite, V-separable models have (finitely many) enough final points. Therefore, the relevant notions are the ones of *ω-hypercanonicity, extensive ω-canonicity, ω-canonicity.* The corresponding definitions are a natural relativisation of the corresponding ones given for canonicity. More precisely:

3.2.1 Definition *Let L be any intermediate logic.*

(a) L is ω-hypercanonical *iff, for every finite V, the underlying frame of any V-separable Kripke model of L^V is a frame for L.*

(b) L is extensively ω-canonical *iff, for every finite V, the underlying frame of any well V-separable Kripke model of L^V is a frame for L.*

□

It is evident that:

- hypercanonicity $\Longrightarrow$ ω-hypercanonicity;
- extensive canonicity $\Longrightarrow$ extensive ω-canonicity.

We also remark that the significant distinction, in which we are mainly interested, is between the extensive ω-canonicity and the "simple" ω-canonicity which requires V-fullness. Note that, since the logic $\mathbf{Bd}_h$ ($h \geq 1$) is ω-hypercanonical and since every V-separable model of $\mathbf{Bd}_h$, for V finite, is finite (see Proposition 1.9.8), we can immediately state that:

3.2.2 Corollary *The logic* $\mathbf{Bd}_h$, *for every* $h \geq 1$, *is characterized by the class of finite frames with depth at most* h. $\square$

We stress that this result implies the decidability of $\mathbf{Bd}_h$.

In next chapters we will provide a more accurate discussion of such notions and we will present general techniques in order to classify logics.

Chapter 4

Analysis of Canonicity

In this chapter we give a systematic study of canonicity, strong completeness and other related notions. The Canonicity Criterion here stated is formally similar to the one in [15], while the Strong Completeness Criterion has a more general formulation than in [15]. Moreover, the techniques here used are different from the ones explained in [15]: as a matter of fact, in the quoted paper an algebraic-categorical approach is adopted, while our techniques, which refer to Kripke semantics, are more inspired to the ones used in Model Theory. We also state a criterion for the hypercanonicity and one for the extensive canonicity. The main result of the chapter is the classification of the logics in one variable with respect to the strong completeness:

- All the intermediate logics axiomatized by axioms in one variable, except four of them, are not strongly complete.

We point out that this significative result has been firstly proved in [15]; before this paper, it was only known that **St** is not strongly complete [33]. Furthermore, we refine this classification, considering also the family of logics in one variable having frames with bounded depth (these results are unpublished). Finally, we give some applications to Medvedev logic and to the logic of rhombuses.

4.1 Some criteria for canonicity

In order to study in a systematic way the notions of canonicity and strong completeness, we introduce the notion of chain of frames and some notions of limit of a chain [1]. A *chain of frames*

$$\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$$

[1] We recall that we take into account only frames with enough final points.

is a sequence of frames $\underline{P}_n = \langle P_n, \leq_n \rangle$, for each $n \geq 1$, and of p-morphisms f_n from $\underline{P}_{n+1}$ onto $\underline{P}_n$. We now define four different notions of limit of a chain. Let $\underline{P} = \langle P, \leq \rangle$ be any frame and let, for each $n \geq 1$, h_n be a p-morphism from $\underline{P}$ onto $\underline{P}_n$.

(1) We say that $\underline{P}$ is a *weak limit* of $\mathscr{C}$ with projections $\{h_n\}_{n \geq 1}$ iff the p-morphisms h_n commute with the p-morphisms f_n, that is:

$$h_n = f_n \circ h_{n+1}, \text{ for every } n \geq 1.$$

This property can be represented by the commutative diagram in Figure 4.1. We

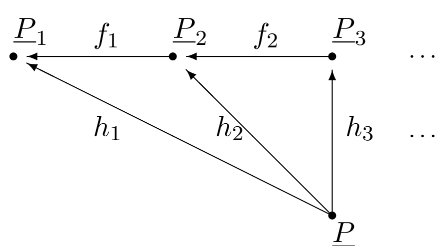

Figure 4.1: Diagram of a chain of frames

call h_n the *projection* of $\underline{P}$ onto $\underline{P}_n$.

(2) We say that $\underline{P}$ is a *separable weak limit* of $\mathscr{C}$ with projections $\{h_n\}_{n \geq 1}$ iff:

- $\underline{P}$ is a weak limit of $\mathscr{C}$ with projections $\{h_n\}_{n \geq 1}$;
- for every $\alpha, \beta \in P$, if, for all $n \geq 1$, $h_n(\alpha) = h_n(\beta)$, then $\alpha = \beta$.

(3) We say that $\underline{P}$ is a *well separable weak limit* of $\mathscr{C}$ with projections $\{h_n\}_{n \geq 1}$ iff:

- $\underline{P}$ is a weak limit of $\mathscr{C}$ with projections $\{h_n\}_{n \geq 1}$;
- for every $\alpha, \beta \in P$, if, for all $n \geq 1$, $h_n(\alpha) \leq_n h_n(\beta)$, then $\alpha \leq \beta$.

(4) We say that $\underline{P}$ is a *limit* of $\mathscr{C}$ with projections $\{h_n\}_{n \geq 1}$ iff:

- $\underline{P}$ is a well separable limit of $\mathscr{C}$ with projections $\{h_n\}_{n \geq 1}$;
- for every $\alpha_1 \in P_1, \ldots, \alpha_n \in P_n \ldots$, if, for all $n \geq 1$, $\alpha_n = f_n(\alpha_{n+1})$, then there is $\alpha \in P$ such that $h_n(\alpha) = \alpha_n$ for every $n \geq 1$.

Clearly, each definition is a proper refinement of the previous one; moreover, the limit of a chain is uniquely determined (up to isomorphisms), as proved in next proposition.

4.1.1 Proposition *Let $\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$ be a chain of frames $\underline{P}_n = \langle P_n, \leq_n\rangle$. Then $\mathscr{C}$ has one and only one limit (up to isomorphisms).*

Proof: To prove the existence of at least one limit, let us define the frame $\underline{P}^* = \langle P^*, \leq^*\rangle$ as follows:

- $P^* = \{\alpha^* = \langle \alpha_1, \alpha_2, \ldots\rangle :$ for every $n \geq 1$, $\alpha_n \in P_n$ and $\alpha_n = f_n(\alpha_{n+1})\}$;
- $\langle \alpha_1, \alpha_2, \ldots\rangle \leq^* \langle \beta_1, \beta_2, \ldots\rangle$ iff, for every $n \geq 1$, $\alpha_n \leq_n \beta_n$.

It is easy to see that $\underline{P}^*$ is a limit of $\mathscr{C}$ having, as projections, the maps h_n^* such that:

$$h_n^*(\langle \alpha_1, \ldots, \alpha_n, \ldots\rangle) = \alpha_n.$$

Suppose now that $\underline{P}'$ and $\underline{P}''$ are two distinct limits of $\mathscr{C}$ with projections $\{h_n'\}_{n\geq 1}$ and $\{h_n''\}_{n\geq 1}$ respectively. Let us define a map g from $\underline{P}'$ to $\underline{P}''$ in the following way:

$$g(\alpha') = \alpha'' \text{ iff } h_n'(\alpha') = h_n''(\alpha'') \text{ for every } n \geq 1 .$$

Then g is an isomorphism between the frames $\underline{P}'$ and $\underline{P}''$, and this completes the proof. □

Hereafter we assume that the limit of a chain is defined as in the proof of the previous proposition. We point out that this kind of construction is typical of Category Theory (see for instance [19]). As a matter of fact, let us consider the category **P** having as objects the posets and as arrows the order preserving maps between posets, with the usual operation of composition. Then a chain of frames $\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$ corresponds to a *cochain diagram*, while a weak limit $\underline{P}$ with projections $\{h_n\}_{n\geq 1}$ corresponds to a *cocone* $\langle \underline{P}, \{h_n\}_{n\geq 1}\rangle$ of $\mathscr{C}$. It is worth noting that the limit $\underline{P}^*$ with the projections $\{h_n^*\}_{n\geq 1}$, as defined in Proposition 4.1.1, is actually a *limit* of $\mathscr{C}$ in the category **P** according to the definition of Category Theory; namely:

(1) $\langle \underline{P}^*, \{h_n^*\}_{n\geq 1}\rangle$ is a cocone of $\mathscr{C}$;

(2) For every other cocone $\langle \underline{P}, \{h_n\}_{n\geq 1}\rangle$ of $\mathscr{C}$, there is a unique order preserving map g from $\underline{P}$ onto $\underline{P}^*$ such that, for every $n \geq 1$, $h_n = h_n^* \circ g$ (*universal property*).

On the other hand, we are not interested in arbitrary order preserving maps, but only in p-morphisms, that is, in maps which preserve the validity of certain formulas. This is the same as restricting to the subcategory $\mathbf{P}'$ of **P**, having as arrows the order preserving maps which are p-morphisms. In this category the limit $\underline{P}^*$ is not even a limit in the categorical sense, since the universal property may lack (indeed, it is not even guaranteed that there is a p-morphism from any weak limit $\underline{P}$ of $\mathscr{C}$ onto $\underline{P}^*$. We now pass to define a chain of Kripke models, which is a natural generalization of the chain of frames. Let $\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$ be a chain of frames and let $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \cdots$ be a sequence of sets of propositional variables such that $\bigcup_{n\geq 1} V_n$ coincides with the set of all the variables of the language. Then

$$\mathscr{C}_K = \{\underline{K}_n, V_n, f_n\}_{n\geq 1}$$

is a *chain of models* if, for every $n \geq 1$, it holds that:

(1) $\underline{K}_n = \langle P_n, \leq_n, \Vdash_n\rangle$ is a V_n-separable model based on the frame $\underline{P}_n$.

(2) f_n is a V_n p-morphism from $\underline{K}_{n+1}$ onto $\underline{K}_n$.

Note that, by (2), for every $\alpha \in P_{n+1}$ we have that:

$$\Gamma^{V_n}_{\underline{K}_{n+1}}(\alpha) = \Gamma^{V_n}_{\underline{K}_n}(f_n(\alpha)).$$

This means that passing from $\alpha' \in P_n$ to any preimage of α' with respect to f_n, the forcing of the V_n-formulas does not change. The chain of frames associated with $\mathscr{C}_K$ is the chain $\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$, where $\underline{P}_n$ is the frame of $\underline{K}_n$.
We can extend the notions of limit of a chain of frames to the case of chains of models. Let $\mathscr{C}_K = \{\underline{K}_n, V_n, f_n\}_{n\geq 1}$ be a chain of models, let $\mathscr{C}$ be the chain of frames associated with $\mathscr{C}_K$, let $\underline{K} = \langle P, \leq, \Vdash\rangle$ be any Kripke model and let $\underline{P} = \langle P, \leq\rangle$ be the frame of $\underline{K}$.

(1) We say that $\underline{K}$ is a *weak limit* of $\mathscr{C}_K$ with projections $\{h_n\}_{n\geq 1}$ iff:

- $\underline{P}$ is a weak limit of $\mathscr{C}$ with projections $\{h_n\}_{n\geq 1}$;
- h_n is a V_n p-morphism from $\underline{K}$ onto $\underline{K}_n$.

(2) We say that $\underline{K}$ is a *separable weak limit* of $\mathscr{C}_K$ with projections $\{h_n\}_{n\geq 1}$ iff:

- $\underline{P}$ is a separable weak limit of $\mathscr{C}$ with projections $\{h_n\}_{n\geq 1}$;
- h_n is a V_n p-morphism from $\underline{K}$ onto $\underline{K}_n$.

(3) We say that $\underline{K}$ is a *well separable weak limit* of $\mathscr{C}_K$ with projections $\{h_n\}_{n\geq 1}$ iff:

- $\underline{P}$ is a well separable weak limit of $\mathscr{C}$ with projections $\{h_n\}_{n\geq 1}$;
- h_n is a V_n p-morphism from $\underline{K}$ onto $\underline{K}_n$.

(4) We say that $\underline{K}$ is a *limit* of $\mathscr{C}_K$ with projections $\{h_n\}_{n\geq 1}$ iff:

- $\underline{P}$ is a limit of $\mathscr{C}$;
- h_n is a V_n p-morphism from $\underline{K}$ onto $\underline{K}_n$.

4.1.2 Proposition *Let* $\mathscr{C}_K = \{\underline{K}_n, V_n, f_n\}_{n\geq 1}$ *be a chain of models and let* $\underline{K} = \langle P, \leq, \Vdash \rangle$ *be a weak limit of* $\mathscr{C}_K$ *having projections* $\{h_n\}_{n\geq 1}$. *Then, for every* $\alpha \in P$, *it holds that:*

(i) $\Gamma^{V_n}_{\underline{K}}(\alpha) = \Gamma^{V_n}_{\underline{K}_n}(h_n(\alpha))$, *for every* $n \geq 1$.

(ii) $\Gamma_{\underline{K}}(\alpha) = \bigcup_{n\geq 1} \Gamma^{V_n}_{\underline{K}_n}(h_n(\alpha))$.

Proof: Since h_n is a V_n p-morphism, (i) immediately follows; (ii) is a consequence of (i). □

Thus, the model $\underline{K}_n$ of a chain $\mathscr{C}_K$ can be viewed as a sort of approximation, up to the V_n-formulas, of any weak limit of $\mathscr{C}_K$.
Let $\mathscr{C}_K$ be a chain of models and let $\underline{P} = \langle P, \leq \rangle$ be a weak limit with projections $\{h_n\}_{n\geq 1}$ of the chain of frames $\mathscr{C}$ associated with $\mathscr{C}_K$; then the weak limit $\underline{K} = \langle P, \leq, \Vdash \rangle$ of $\mathscr{C}_K$ based on $\underline{P}$ and having the same projections $\{h_n\}_{n\geq 1}$ is uniquely determined by the following condition:

for every $p \in V_n$, $\alpha \Vdash p$ iff $h_n(\alpha) \Vdash_n p$

One can easily check that the above condition actually defines a forcing relation; in particular, the limit of a chain of models is unique, up to isomorphisms. Note that, choosing different projections, we obtain different forcing relations. We now study the properties of the models of $\mathscr{C}_K$ which are preserved in weak limits.

4.1.3 Proposition *Let* L *be an intermediate logic and let* $\mathscr{C}_K = \{\underline{K}_n, V_n, f_n\}_{n\geq 1}$ *be a chain of models* $\underline{K}_n$ *of* L. *Then every weak limit of* $\mathscr{C}_K$ *is a model of* L.

Proof: Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a weak limit of $\mathscr{C}_K$ having projections $\{h_n\}_{n\geq 1}$. Let us take any formula A of L and any point α of $\underline{P}$. Let $n \geq 1$ be such that $\mathrm{Var}(A) \subseteq V_n$; since $h_n(\alpha) \Vdash_n A$ and $\Gamma^{V_n}_{\underline{K}}(\alpha) = \Gamma^{V_n}_{\underline{K}_n}(h_n(\alpha))$, it follows that $\alpha \Vdash A$, hence $\underline{K}$ is a model of L. □

4.1.4 Proposition *Let* L *be an intermediate logic and let* $\mathscr{C}_K = \{\underline{K}_n, V_n, f_n\}_{n\geq 1}$ *be a chain of models* $\underline{K}_n$ *of* L. *Then every separable weak limit of* $\mathscr{C}_K$ *is a separable model of* L.

Proof: Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a separable weak limit of $\mathscr{C}_K$ having projections $\{h_n\}_{n\geq 1}$. By Proposition 4.1.3, $\underline{K}$ is a model of L. To prove the separability, let $\alpha, \beta \in P$ be such that $\Gamma_{\underline{K}}(\alpha) = \Gamma_{\underline{K}}(\beta)$. Then, for every $n \geq 1$, $\Gamma^{V_n}_{\underline{K}}(\alpha) = \Gamma^{V_n}_{\underline{K}}(\beta)$, hence $\Gamma^{V_n}_{\underline{K}_n}(h_n(\alpha)) = \Gamma^{V_n}_{\underline{K}_n}(h_n(\beta))$. Since, by definition of $\mathscr{C}_K$, $\underline{K}_n$ is V_n-separable, we have that $h_n(\alpha) = h_n(\beta)$ for every $n \geq 1$; by definition of separable weak limit, it follows that $\alpha = \beta$. □

4.1.5 Proposition *Let* L *be an intermediate logic and let* $\mathscr{C}_K = \{\underline{K}_n, V_n, f_n\}_{n\geq 1}$ *be a chain of well* V_n*-separable models* $\underline{K}_n$ *of* L. *Then every well separable weak limit of* $\mathscr{C}_K$ *is a well separable model of* L.

Proof: Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a well separable weak limit of $\mathscr{C}_K$ having projections $\{h_n\}_{n\geq 1}$. By Proposition 4.1.3, $\underline{K}$ is a model of L. To prove the well separability, let $\alpha, \beta \in P$ be such that $\Gamma_{\underline{K}}(\alpha) \subseteq \Gamma_{\underline{K}}(\beta)$. Then, for every $n \geq 1$, $\Gamma^{V_n}_{\underline{K}}(\alpha) \subseteq \Gamma^{V_n}_{\underline{K}}(\beta)$, hence $\Gamma^{V_n}_{\underline{K}_n}(h_n(\alpha)) \subseteq \Gamma^{V_n}_{\underline{K}_n}(h_n(\beta))$. Since $\underline{K}_n$ is well V_n-separable, we have that $h_n(\alpha) \leq_n h_n(\beta)$ for every $n \geq 1$; by definition of well separable weak limit, it follows that $\alpha \leq \beta$. □

4.1.6 Proposition *Let L be an intermediate logic and let $\mathscr{C}_K = \{\underline{K}_n, V_n, f_n\}_{n\geq 1}$ be a chain of V_n-full models $\underline{K}_n$ of L. Then the limit $\underline{K}^* = \langle P^*, \leq^*, \Vdash^* \rangle$ of $\mathscr{C}_K$ is a separable and full model of L.*

Proof: By Proposition 4.1.4, $\underline{K}^*$ is a separable model of L. To prove the fullness, let $\alpha^* = \langle \alpha_1, \alpha_2, \ldots \rangle$ be a point of $\underline{K}^*$ and let Δ be any saturated set such that $\Gamma_{\underline{K}^*}(\alpha^*) \subseteq \Delta$. Let, for every $n \geq 1$, Δ_n be the set of all the V_n-formulas of Δ. Then, for every $n \geq 1$, $\Gamma^{V_n}_{\underline{K}^*}(\alpha^*) \subseteq \Delta_n$, that is $\Gamma^{V_n}_{\underline{K}_n}(\alpha_n) \subseteq \Delta_n$. Since Δ_n is a V_n-saturated set and $\underline{K}_n$ is V_n-full, there is $\beta_n \in P_n$ such that $\alpha_n \leq_n \beta_n$ and $\Gamma^{V_n}_{\underline{K}_n}(\beta_n) = \Delta_n$; moreover, since $\Gamma^{V_n}_{\underline{K}_{n+1}}(\beta_{n+1}) = \Delta_n = \Gamma^{V_n}_{\underline{K}_n}(\beta_n)$ and $\underline{K}_n$ is V_n-separable, it holds that $\beta_n = f_n(\beta_{n+1})$. Therefore, $\beta^* = \langle \beta_1, \ldots, \beta_n, \ldots \rangle$ is a point of $\underline{P}^*$ such that $\alpha^* \leq^* \beta^*$. We have:

$$\Gamma_{\underline{K}^*}(\beta^*) = \bigcup_{n\geq 1} \Gamma^{V_n}_{\underline{K}_n}(\beta_n) = \bigcup_{n\geq 1} \Delta_n = \Delta$$

and this proves the fullness of $\underline{K}^*$. □

This proposition can be used to build "big" full models. Indeed, when we are concerned with finite models, no problems arise, since finite models are also full. On the other hand, when we deal with an infinite model $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$, it is not trivial to check that $\underline{K}$ contains points in correspondence of all the saturated sets containing $\Gamma_{\underline{K}}(\rho)$. The previous proposition allows us to get over the difficulty, at least when the full model $\underline{K}$ can be approximated by means of finite models $\underline{K}_n$. To complete the picture, we point out that weak limits (and also limits) do not preserve, in general, the *first-order* properties valid in all the frames of a chain, as we will see later.

Now we formulate some criteria for hypercanonicity, extensive canonicity and canonicity respectively.

4.1.7 Theorem (Hypercanonicity Criterion)
Let L be an hypercanonical logic of and let $\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$ be a chain of frames $\underline{P}_n = \langle P_n, \leq_n \rangle$ for L such that P_n is countable. Then every separable weak limit of $\mathscr{C}$ is a frame for L.

Proof: We can define a chain of models

$$\mathscr{C}_K = \{\underline{K}_n, V_n, f_n\}_{n\geq 1}$$

where $\underline{K}_n$ is based on $\underline{P}_n$ and $\underline{K}_n$ is V_n-separable (or even well V_n-separable). Indeed, since the frames involved are countable, we can choose an increasing sequence of countable sets V_n (contained in the countable set of all the variables of the language) such that any two points of $\underline{P}_n$ are (well) separated by some propositional variable of V_n. Moreover, since $\underline{P}_n$ is a frame for L, $\underline{K}_n$ is a model of L. Let $\underline{P}$ be a separable weak limit of $\mathscr{C}$ having projections $\{h_n\}_{n\geq 1}$ and let $\underline{K}$ be the separable weak limit model of $\mathscr{C}_K$ based on $\underline{P}$ and having the same projections. By Proposition 4.1.4, $\underline{K}$ is a separable model of L; by the hypercanonicity of L, we can conclude that $\underline{P}$ is a frame for L. □

4.1.8 Theorem (Extensive Canonicity Criterion)
Let L be an extensive canonical logic and let $\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$ be a chain of frames $\underline{P}_n = \langle P_n, \leq_n\rangle$ for L such that P_n is countable. Then every well separable weak limit of $\mathscr{C}$ is a frame for L.

Proof: As in the proof of the Hypercanonicity Criterion, we can define a chain of models

$$\mathscr{C}_K = \{\underline{K}_n, V_n, f_n\}_{n\geq 1}$$

where $\underline{K}_n$ is based on $\underline{P}_n$ and $\underline{K}_n$ is well V_n-separable. By Proposition 4.1.5, $\underline{K}$ is a well separable model of L; by the extensive canonicity of L, $\underline{P}$ is a frame for L. □

We now state the Canonicity Criterion, which essentially coincides with the formulation in [15].

4.1.9 Theorem (Canonicity Criterion)
Let L be a canonical logic and let $\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$ be a chain of finite frames $\underline{P}_n = \langle P_n, \leq_n\rangle$ for L. Then the limit of $\mathscr{C}$ is a frame for L.

Proof: Let us define a chain of models

$$\mathscr{C}_K = \{\underline{K}_n, V_n, f_n\}_{n\geq 1}$$

where $\underline{K}_n$ is a V_n-separable model based on $\underline{P}_n$ (note that we only need an increasing sequence of finite sets V_n). Since $\underline{P}_n$ is a finite frame for L, it follows that $\underline{K}_n$ is also a V_n-full model of L. By Proposition 4.1.6, the limit $\underline{K}^*$ of $\mathscr{C}_K$ is a separable and full model of L; since L is canonical, the frame $\underline{P}^*$ of $\underline{K}^*$ (that is, the limit of $\mathscr{C}$) is a frame for L. □

We remark that the limit $\underline{P}^*$ in the proof of the Canonicity Criterion is actually a generated subframe of the frame of the canonical model of L. We also stress that it is not immediate to extend the Canonicity Criterion to chains of countable frames; to do this, we have to give suitable conditions on the frame $\underline{P}_n$, in order to define a full model $\underline{K}_n$ on $\underline{P}_n$. One could exploit, for instance, the arguments explained in next chapter; on the other hand, we have not in mind significative applications of such an extended criterion.

4.2 A criterion for the strong completeness

We now pass to the analysis of strong completeness. As the definition suggests, we have to consider *all* the models which realize any L-saturated set Δ. This requires a deeper study of the weak limits of a chain and of the relations between weak limits and the limit, which is, roughly speaking, the "biggest" model of Δ. In the following proposition we show that, in some cases, we can completely characterize all the (non necessarily separable) models of Δ.

4.2.1 Proposition *Let $\mathscr{C}_K = \{\underline{K}_n, V_n, f_n\}_{n\geq 1}$ be a chain of finite models $\underline{K}_n = \langle P_n, \leq_n, \rho_n, \Vdash_n\rangle$, let $\Delta = \bigcup_{n\geq 1}\Gamma^{V_n}_{\underline{K}_n}(\rho_n)$ and let $\underline{K} = \langle P, \leq, \rho, \Vdash\rangle$ be any Kripke model. Then, $\Gamma_{\underline{K}}(\rho) = \Delta$ if and only if $\underline{K}$ is a weak limit of $\mathscr{C}_K$.*

Proof: The "if" part corresponds to Point (ii) of Proposition 4.1.2. Suppose now that $\Gamma_{\underline{K}}(\rho) = \Delta$ and let us define, for each $n \geq 1$, a map h_n from the points of $\underline{K}$ to the points of $\underline{K}_n$ in the following way:

$$h_n(\alpha) = \alpha' \text{ iff } \Gamma^{V_n}_{\underline{K}}(\alpha) = \Gamma^{V_n}_{\underline{K}_n}(\alpha').$$

Since, for each $n \geq 1$, $\Gamma^{V_n}_{\underline{K}}(\rho) = \Gamma^{V_n}_{\underline{K}_n}(\rho_n)$ and $\underline{K}_n$ is a finite V_n-separable model, we can apply Proposition 1.9.5 and claim that h_n is a V_n p-morphism from $\underline{K}$ onto $\underline{K}_n$. Moreover, by definition of $\mathscr{C}_K$ the maps h_n commute with the maps f_n; this means that $\underline{K}$ is a weak limit of $\mathscr{C}_K$ with projections $\{h_n\}_{n\geq 1}$. □

4.2.2 Theorem (Necessary Condition for Strong Completeness)
Let L be a strongly complete logic and let $\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$ be a chain of finite frames $\underline{P}_n = \langle P_n, \leq_n, \rho_n\rangle$ for L. Then, there is a weak limit $\underline{P} = \langle P, \leq, \rho\rangle$ of $\mathscr{C}$ which is a frame for L.

Proof: As in the proof of the Canonicity Criterion, we can define a chain of models $\mathscr{C}_K = \{\underline{K}_n, V_n, f_n\}_{n\geq 1}$, where the V_n-separable model $\underline{K}_n$ is based on the frame $\underline{P}_n$. Let us consider the L-saturated set $\Delta = \bigcup_{n\geq 1}\Gamma^{V_n}_{\underline{K}_n}(\rho_n)$; since L is strongly complete, there must be a model $\underline{K} = \langle P, \leq, \rho, \Vdash\rangle$ such that $\Gamma_{\underline{K}}(\rho) = \Delta$ and $\underline{P} = \langle P, \leq, \rho\rangle$ is a frame for L. Since the models $\underline{K}_n$ are finite, we can apply Proposition 4.2.1 and claim that $\underline{K}$ is a weak limit of $\mathscr{C}_K$, hence $\underline{P}$ is a weak limit of $\mathscr{C}$. □

This theorem is not of great use if our concern is to disprove the strong completeness of L; indeed, we should check that *all* the weak limits of the chain $\mathscr{C}$ are not frames for L. On the other hand, we can limit ourselves to study particular frames, namely the stable reductions of the limit of $\mathscr{C}$, which convey useful information about weak limits.

Let $\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$ be a chain of frames and let $\underline{P}^* = \langle P^*, \leq^*\rangle$ be the limit of $\mathscr{C}$. We say that $\alpha^* \in P^*$ is *stable* if we definitively (i.e., for all n greater then some integer k) have that α_n has only one preimage with respect to f_n.

4.2.3 Proposition *Let $\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$ be a chain of frames $\underline{P}_n = \langle P_n, \leq_n, \rho_n\rangle$, let $\underline{P}^* = \langle P^*, \leq^*, \rho^*\rangle$ be the limit of $\mathscr{C}$ and let $\underline{P} = \langle P, \leq, \rho\rangle$ be a weak limit of $\mathscr{C}$. Then there is a map $h : P \to P^*$ such that:*

(i) $h(\rho) = \rho^*$;

(ii) $\alpha \leq \beta$ *implies* $h(\alpha) \leq^* h(\beta)$;

(iii) if $h(\alpha) <^* \beta^*$ *and* β^* *is stable, then there is* $\beta \in P$ *such that* $\alpha < \beta$ *and* $h(\beta) = \beta^*$.

Proof: Suppose that $\underline{P}$ is a weak limit of $\mathscr{C}$ having projections $\{h_n\}_{n\geq 1}$ and let us define, for each $\alpha \in P$:

$$h(\alpha) = \langle h_1(\alpha), h_2(\alpha), \ldots\rangle.$$

By definition, $h_n(\alpha) = f_n(h_{n+1}(\alpha))$ for every $n \geq 1$, hence $h(\alpha)$ is a point of $\underline{P}^*$ and h is a map from $\underline{P}$ to $\underline{P}^*$. It is immediate to prove that $h(\rho) = \langle \rho_1, \rho_2, \ldots\rangle = \rho^*$ and that h is order preserving. Suppose now that $h(\alpha) <^* \beta^*$ (i.e. $h_n(\alpha) \leq_n \beta_n$ for every $n \geq 1$) and that $\beta^* = \langle \beta_1, \beta_2, \ldots\rangle$ is stable. Then there is n such that, for every $k \geq n$, β_{k+1} is the only preimage of β_k with respect to f_k. Since $h_n(\alpha) <_n \beta_n$, there is $\beta \in P$ such that $\alpha < \beta$ and $h_n(\beta) = \beta_n$; by induction on j, we can prove that $h_j(\beta) = \beta_j$ for every $j \geq n$. This also implies that $h_j(\beta) = \beta_j$ for every $j \geq 1$, thus:

$$h(\beta) = \langle h_1(\beta), h_2(\beta), \ldots\rangle = \langle \beta_1, \beta_2, \ldots\rangle = \beta^*$$

and (iii) is proved. □

We remark that the map h of the previous proposition is not, in general, a p-morphism, since the "openness" property is guaranteed only for the stable points of $\underline{P}^*$ (thus, in general, it may be not even surjective).
Now we show that the stable points of a full model of a logic have a primary importance in determining the strong completeness. To this aim, we introduce the following definition.

4.2.4 Definition *Let $\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$ be a chain of frames and let $\underline{P}^* = \langle P^*, \leq^*\rangle$ be the limit of $\mathscr{C}$. We say that $\underline{P} = \langle P, \leq\rangle$ is a* stable reduction *of $\underline{P}^*$ iff there is a p-morphism g from $\underline{P}^*$ onto $\underline{P}$ such that, for every $\alpha^* \in P^*$ and every $\beta \in P$, the following holds:*

- *if $g(\alpha^*) < \beta$, then there is $\beta^* \in P^*$ s.t. $\alpha^* <^* \beta^*$, β^* is stable and $g(\beta^*) = \beta$.*

□

4.2.5 Proposition *Let $\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$ be a chain of frames $\underline{P}_n = \langle P_n, \leq_n, \rho_n\rangle$, let $\underline{P} = \langle P, \leq, \rho\rangle$ be a weak limit of $\mathscr{C}$ and let $\underline{P}' = \langle P', \leq', \rho'\rangle$ be a stable reduction of the limit $\underline{P}^*$ of $\mathscr{C}$. Then there is a p-morphism f from $\underline{P}$ onto $\underline{P}'$.*

Proof: Let $\underline{P}^* = \langle P^*, \leq^*, \rho^* \rangle$ be the limit of $\mathscr{C}$, let $h : P \to P^*$ be the map defined in Proposition 4.2.3 and let $g : P^* \to P'$ be as in the definition of stable reduction. We know that h is "almost" a p-morphism, while g is "much more" than a p-morphism; we show that the composite map $f = g \circ h$ is a p-morphism. It is immediate to prove that f is order preserving. Suppose now that $g(h(\alpha)) <' \beta'$, for some $\alpha \in P$ and $\beta' \in P'$. By definition of g, there is $\beta^* \in P^*$ such that β^* is stable, $h(\alpha) <^* \beta^*$ and $g(\beta^*) = \beta'$. By definition of h, there is $\beta \in P$ such that $\alpha < \beta$ and $h(\beta) = \beta^*$, hence $g(h(\beta)) = \beta'$. Finally, $g(h(\rho)) = g(\rho^*) = \rho'$, thus f is also surjective and the proposition is proved. □

This can be depicted by the commutative diagram in Figure 4.2, where we have put into evidence the arrows which represent p-morphisms. It follows that any stable

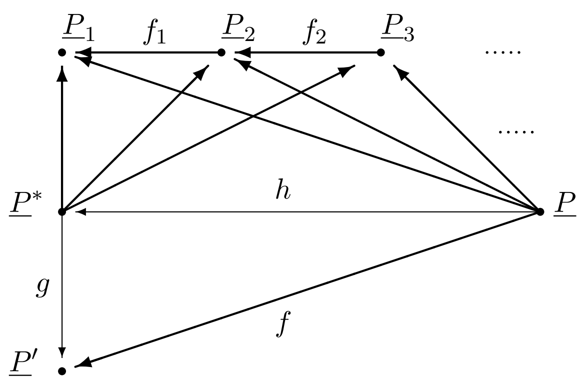

Figure 4.2: Diagram of weak limits

reduction of the limit of $\mathscr{C}$ is representative, in some sense, of all the weak limits of $\mathscr{C}$; thus, our Necessary Condition for Strong Completeness can be reformulated in the following more interesting form.

4.2.6 Theorem (Strong Completeness Criterion)
Let L be a strongly complete logic, let $\mathscr{C} = \{\underline{P}_n, f_n\}_{n \geq 1}$ be a chain of finite frames $\underline{P}_n = \langle P_n, \leq_n, \rho_n \rangle$ for L, and let $\underline{P}' = \langle P', \leq', \rho' \rangle$ be a stable reduction of the limit $\underline{P}^*$ of $\mathscr{C}$. Then $\underline{P}'$ is a frame for L.

Proof: Since L is strongly complete, by the Necessary Condition for Strong Completeness there must be a weak limit $\underline{P} = \langle P, \leq, \rho \rangle$ of $\mathscr{C}$ such that $\underline{P}$ is a frame for L. By Proposition 4.2.5, there is a p-morphism from $\underline{P}$ onto $\underline{P}'$; since $\underline{P}$ is a frame for L, we can conclude that also $\underline{P}'$ is a frame for L. □

Thus, to disprove the strong completeness of a logic, we can restrict ourselves to study the stable reductions of the limits. We also point out that the previous criterion is more general than the corresponding one explained in [15].

4.3 Strong completeness of the logics in one variable

As an applications of these criteria, we give a detailed classification of the logics in one variable with respect to canonicity and strong completeness.

4.3.1 The canonical logics in one variable

The first four logics in our enumeration turn out to be canonical, even better, they are the only canonical logics in the class. The case of classical logic **Cl** is trivial, while we have already seen that Jankov logic **Jn** is hypercanonical (more precisely, as proved in Theorem 3.1.7, it is $QHYP_2$).

We now analyze the remaining ones.

4.3.1 Theorem *The logic* $\mathbf{NL}_{3,4}$ *is hypercanonical.*

Proof: Suppose, by absurd, that $\mathbf{NL}_{3,4}$ is not hypercanonical; then there is a separable model (with enough final points) $\underline{K} = \langle P, \leq, \Vdash \rangle$ of $\mathbf{NL}_{3,4}$ whose frame $\underline{P} = \langle P, \leq \rangle$ is not a frame for $\mathbf{NL}_{3,4}$. Then there are $\alpha, \beta, \varphi_1, \varphi_2$ in $\underline{P}$ such that:

- $\alpha < \beta$, $\alpha < \varphi_2$, $\beta < \varphi_1$, and $\beta \not< \varphi_2$.

Without loss of generality (by definition of p-morphism) we can assume that φ_1 and φ_2 are final points; thus, by the hypothesis of separability of $\underline{K}$, there exists a formula A such that:

- $\beta \Vdash A$ and $\varphi_2 \nVdash A$ (hence $\varphi_2 \Vdash \neg A$).

Moreover, since β is not final, there must be a formula B such that:

- $\beta \Vdash \neg\neg B$ and $\beta \nVdash B$.

Let us take the formula $H = \neg\neg A \wedge B$. Then:

- $\varphi_2 \Vdash \neg H$;

- $\beta \Vdash \neg\neg H$ and $\beta \nVdash H$.

This implies that $\alpha \nVdash \neg\neg H \vee (\neg\neg H \to H)$, which is an instance of $\boldsymbol{nf}_{3,4}$, a contradiction. Thus the initial assumption is false and $\mathbf{NL}_{3,4}$ is hypercanonical. □

Note that we can prove, in the same way, that $\mathbf{NL}_{3,4}$ is $QHYP_1$; as a matter of fact, if we assume that $\underline{K}$ is well separable, it is not necessary to assume that φ_1 and φ_2 are final.

To prove the canonicity of the next logic, that is the logic $\mathbf{NL}_{4,5}$, we need additional properties on the models.

4.3.2 Theorem *The logic* $\mathbf{NL}_{4,5}$ *is extensively canonical.*

Proof: Suppose that, by absurd, such a logic is not extensively canonical; then there is a well separable model $\underline{K} = \langle P, \leq, \Vdash \rangle$ of $\mathbf{NL}_{4,5}$ whose frame $\underline{P} = \langle P, \leq \rangle$ is not a frame for $\mathbf{NL}_{4,5}$. This implies that there are some points $\alpha, \beta, \varphi_1, \varphi_2$ of $\underline{P}$ such that:

- $\alpha < \beta$ and $\alpha < \gamma$;
- β and γ are two non-final points such that $\beta \not\leq \gamma$ and $\gamma \not\leq \beta$;
- φ_1 and φ_2 are two distinct final points such that $\gamma < \varphi_1$, $\gamma < \varphi_2$ and $\beta \not\leq \varphi_2$.

By the well separability of $\underline{K}$, there are some formulas A, B, C such that:

- $\beta \Vdash A$ and $\varphi_2 \nVdash A$ (hence $\varphi_2 \Vdash \neg A$);
- $\varphi_1 \Vdash B$ and $\varphi_2 \nVdash B$ (hence $\varphi_2 \Vdash \neg B$);
- $\gamma \Vdash C$ and $\beta \nVdash C$.

Moreover, since β is not final, there is a formula D such that:

- $\beta \Vdash \neg\neg D$ and $\beta \nVdash D$.

Let us take the formula $H = \neg\neg(A \vee B) \wedge (D \vee C)$. Then:

- $\beta \nVdash \neg\neg H \to H$ (indeed, $\beta \Vdash \neg\neg H$ and $\beta \nVdash H$);
- $\gamma \Vdash \neg\neg H \to H$;
- $\gamma \nVdash \neg H$ (indeed, $\varphi_1 \Vdash H$);
- $\gamma \nVdash H$ (indeed, $\varphi_2 \Vdash \neg H$).

It follows that $\alpha \nVdash (\neg\neg H \to H) \vee ((\neg\neg H \to H) \to H \vee \neg H)$, which is an instance of $\boldsymbol{nf}_{4,5}$, a contradiction. This means that the initial hypothesis is false, hence $\mathbf{NL}_{4,5}$ is extensively canonical. □

We show that, in the previous proof, the hypothesis of well separability (used to separate γ from β) is essential.

4.3.3 Theorem *The logic* $\mathbf{NL}_{4,5}$ *is not hypercanonical.*

Proof: Let us take a chain of frames $\mathscr{C} = \{\underline{P}_n, f_n\}_{n \geq 1}$, where $\underline{P}_n = \langle P_n, \leq_n, r \rangle$, for each $n \geq 1$, is the frame defined as follows:

- $P_n = \{r, a, b_1, \ldots, b_n, \beta, c, d\}$;
- The ordering relation $\leq_n$ is defined as in Figure 4.3.

The p-morphism f_n from $\underline{P}_{n+1}$ onto $\underline{P}_n$ is defined as follows:

- $f_n(b_{n+1}) = \beta$;

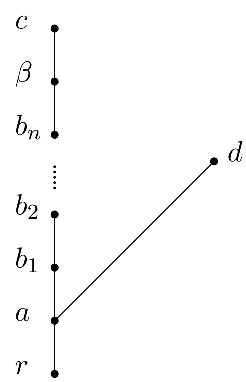

Figure 4.3: The frame $\underline{P}_n$ for $\mathbf{NL}_{4,5}$.

- $f_n(\delta) = \delta$ for all the other points δ.

It is immediate to check that the frames $\underline{P}_n$ are frames for $\mathbf{NL}_{4,5}$. Let us consider the infinite frame $\underline{P} = \langle P, \leq, r \rangle$, where:

- $P = \{r, a, b_1, b_2, \ldots, \beta, c, d\}$;

- The ordering relation $\leq$ is defined as in Figure 4.4.

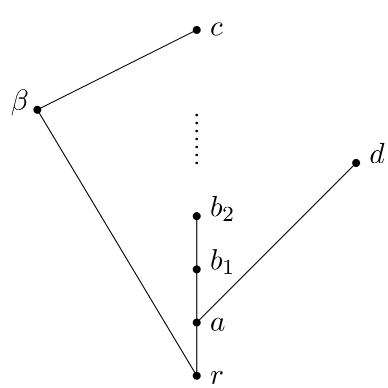

Figure 4.4: The well separable weak limit $\underline{P}$

We point out that, for every $n \geq 1$, $b_n < c$ and $b_n \not< \beta$. It is easy to prove that $\underline{P}$ is a separable weak limit of $\mathscr{C}$ with projections $\{h_n\}_{n\geq 1}$ defined as follows:

- $h_n(b_k) = \beta$ for every $k \geq n+1$;

- $h_n(\delta) = \delta$ for all the other points δ.

Clearly $\underline{P}$ is not a frame for $\mathbf{NL}_{4,5}$; by the Hypercanonicity Criterion, we can conclude that $\mathbf{NL}_{4,5}$ is not hypercanonical. □

We observe that, in the previous proof, the frame $\underline{P}$ is not a well separable weak limit of $\mathscr{C}$ (for instance, it holds that $h_n(a) \leq_n h_n(\beta)$ for every $n \geq 1$, while it is not true that $a \leq \beta$), as it is expected by the fact that $\mathbf{NL}_{4,5}$ is extensively canonical and by the Extensive Canonicity Criterion. To obtain a well separable weak limit with the same projections, we have to put β over all the points b_n (clearly, the frame so obtained is a frame for $\mathbf{NL}_{4,5}$).

4.3.2 The Scott logic St

We have exhausted the examination of the canonical logics in one variable and we pass to study the infinitely many non canonical ones. We begin with the logic $\mathbf{NL}_6 = \mathbf{St}$.

4.3.4 Theorem *The logic* **St** *is not strongly complete.*

Proof: We show a chain $\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$ of finite frames $\underline{P}_n = \langle P_n, \leq_n, r\rangle$ for **St** such that the frame $\underline{P}_{\sigma_5}$ is a stable reduction of the limit $\underline{P}^* = \langle P^*, \leq^*, r^*\rangle$ of $\mathscr{C}$[2]. Let $\underline{P}_n$ be defined as follows:

- $P_n = \{r, a_1, \ldots, a_n, \alpha, b, d_1, \ldots, d_n, \delta\}$;

- the ordering relation $\leq_n$ is defined as in Figure 4.5.

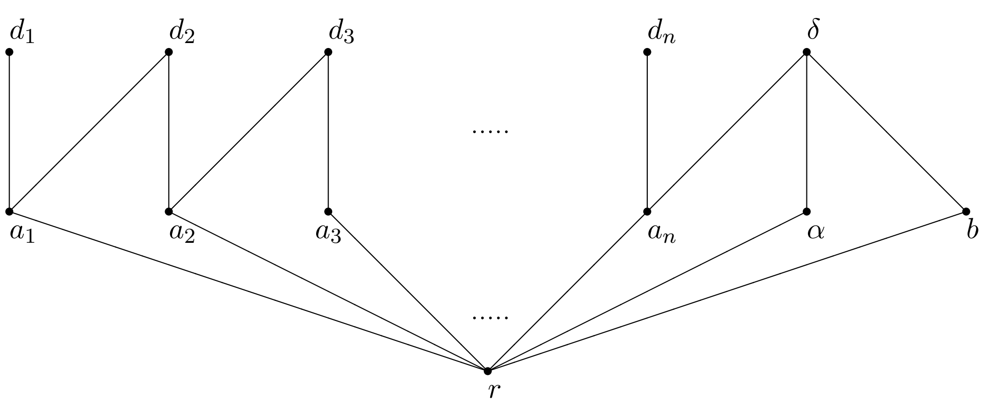

Figure 4.5: The frame $\underline{P}_n$ for **St**

The p-morhpism f_n from $\underline{P}_{n+1}$ onto $\underline{P}_n$ is defined as follows:

[2]Such a chain is similar to the one in [15].

- $f_n(a_{n+1}) = \alpha$;

- $f_n(d_{n+1}) = \delta$;

- $f_n(\beta) = \beta$ for all the other points β.

The limit model $\underline{P}^*$ contains the stable points $r_n^* = \langle r, r, \ldots \rangle$, $a_n^* = \langle \alpha, \ldots, \alpha, a_n, a_n, \ldots \rangle$ (where α occurs in the first $n-1$ components and a_n in the remaining ones), $b^* = \langle b, b, \ldots \rangle$, $d_n^* = \langle \delta, \ldots, \delta, d_n, d_n, \ldots \rangle$, and the non-stable points $\alpha^* = \langle \alpha, \alpha, \ldots \rangle$ and $\delta^* = \langle \delta, \delta, \ldots \rangle$; the ordering relation between these points is described by Figure 4.6. Finally, let g be the p-morphism from $\underline{P}^*$ onto $\underline{P}_{\sigma_5}$

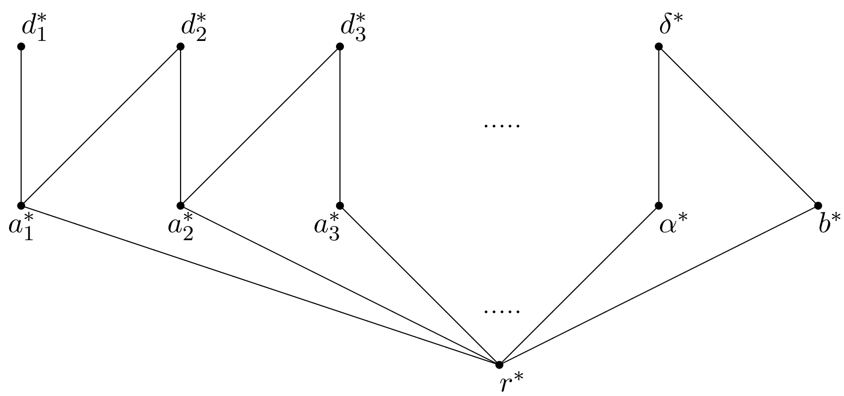

Figure 4.6: The limit frame $\underline{P}^*$

defined as follows:

- $g(d_n^*) = \sigma_1$, for every $n \geq 1$;

- $g(\alpha^*) = g(b^*) = g(\delta^*) = \sigma_2$;

- $g(a_n^*) = \sigma_3$, for every $n \geq 1$;

- $g(r^*) = \sigma_5$.

By definition of g, $\underline{P}_{\sigma_5}$ is a stable reduction of $\underline{P}^*$; we can apply the Strong Completeness Criterion and claim that **St** is not strongly complete. □

We point out that, if we are only interested in disproving the canonicity of **St**, we can limit ourselves to observe that the limit $\underline{P}^*$ of $\mathscr{C}$ is not a frame for **St** and then apply the Canonicity Criterion. We take advantage of this example to observe that, in general, the limit of a chain $\mathscr{C}$ does not inherit the *first-order properties* which hold in all frames $\underline{P}_n$ of $\mathscr{C}$. As a matter of fact, in all $\underline{P}_n$ there is a final point, that is δ, which is an immediate successor of three distinct points of $\underline{P}_n$, and this can be

expressed by a first-order sentence; on the other hand, the limit $\underline{P}^*$ of $\mathscr{C}$ does not enjoy this property.
Finally, we observe that all the frames involved in the previous proof have depth 3; thus we can trivially extend the previous result to the family of logics $\mathbf{St}+\ \mathbf{Bd}_h$, with $h \geq 3$, having, as frames, the frames for $\mathbf{St}$ with depth at most h. More precisely:

4.3.5 Theorem

(i) The logics $\mathbf{St}+\ \mathbf{Bd}_h$*, for* $h < 3$*, are hypercanonical.*

(ii) The logics $\mathbf{St}+\ \mathbf{Bd}_h$*, for* $h \geq 3$*, are not strongly complete.*

Proof:
(i) Let $\underline{K} = \langle P, \leq, \Vdash\rangle$ be a separable model of $\mathbf{St}+\ \mathbf{Bd}_h$, with $h < 3$. Then the frame $\underline{P}$ of $\underline{K}$ has depth at most 2, otherwise an instance of $\boldsymbol{bd}_h$ is not valid in $\underline{K}$; this implies that all the final points of $\underline{P}$ are prefinally connected, thus $\underline{P}$ is also a frame for $\mathbf{St}$.
(ii) It is proved as Theorem 4.3.4, observing that the frames of the chain $\mathscr{C}$ are frames for the logic $\mathbf{St}+\ \mathbf{Bd}_h$. □

4.3.3 The logics $\mathbf{NL}_{m+1}$ $(m \geq 7)$ and $\mathbf{NL}_{n+1,n+2}$ $(n \geq 4)$

By the fact that the frames $\underline{P}_{\sigma_m}$ and $\underline{P}_{\sigma_{n,n+1}}$, for $m \geq 7$ and $n \geq 4$, contain $\underline{P}_{\sigma_5}$ as generated subframe, we can extend without great effort the proof of non strong completeness of $\mathbf{St}$ to the logics $\mathbf{NL}_{m+1}$ and $\mathbf{NL}_{n+1,n+2}$ (namely, the logics in one variable strictly included in $\mathbf{St}$).

4.3.6 Theorem

(i) The logics $\mathbf{NL}_{m+1}$*, for* $m \geq 7$*, are not strongly complete.*

(ii) The logics $\mathbf{NL}_{m+1,m+2}$*, for* $m \geq 4$*, are not strongly complete.*

Proof:
(i) Let $m \geq 7$; we define a chain $\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$ of finite frames $\underline{P}_n = \langle P_n, \leq_n, t_m\rangle$ for $\mathbf{NL}_{m+1}$ such that $\underline{P}_{\sigma_m}$ is a stable reduction of the limit $\underline{P}^* = \langle P^*, \leq^*, t^*_m\rangle$ of $\mathscr{C}$. Let $\underline{P}'_n = \langle P'_n, \leq', t_5\rangle$ be the frame defined as the frame $\underline{P}_n$ in the proof of Theorem 4.3.4, where t_5 coincides with r; then the frame $\underline{P}_n$ is defined as in Figure 4.7. The p-morphism f_n is defined as in the proof of Theorem 4.3.4 on the points $\beta > t_5$ and $f_n(\beta) = \beta$ if β is one of the points t_k. Finally, the limit frame $\underline{P}^*$ of $\mathscr{C}$ is defined as in Figure 4.8, where $\underline{P}'^*$ coincides with the limit frame in the proof of Theorem 4.3.4. It is not difficult to prove that $\underline{P}_{\sigma_m}$ is a stable reduction of $\underline{P}^*$, therefore, by the Strong Completeness Criterion, $\mathbf{NL}_{m+1}$ is not strongly complete.
(ii) We can proceed as in (i) taking, as frame $\underline{P}_n = \langle P_n, \leq_n, t\rangle$ for the logic $\mathbf{NL}_{m+1,m+2}$ $(m \geq 4)$, the one in Figure 4.9 The p-morphisms f_n and the limit $\underline{P}^* = \langle P^*, \leq^*, t^*\rangle$ are defined similarly to in (i). Let us consider the frame $\underline{P}_{\tilde{\sigma}_m}$ in Figure 4.10. It is immediate to prove that $\underline{P}_{\tilde{\sigma}_m}$ is a stable reduction of $\underline{P}^*$ and

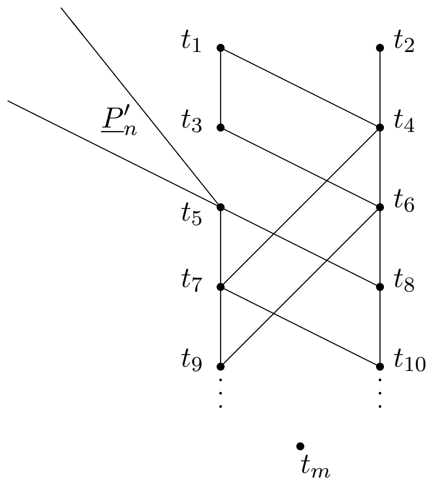

Figure 4.7: The frame $\underline{P}_n$ for $\mathbf{NL}_{m+1}$

that $\underline{P}_{\tilde{\sigma}_m}$ is not a frame for $\mathbf{NL}_{m+1,m+2}$. We can conclude that $\mathbf{NL}_{m+1,m+2}$ is not strongly complete. □

As in the proof relative to **St**, the frames used in the previous proof have minimal depth, in the sense of next theorem.

4.3.7 Theorem
(i) Let $m \geq 7$ and let $h_m = depth(\underline{P}_{\sigma_m})$. Then:

- *for $1 \leq h < h_m$, $\mathbf{NL}_{m+1} +$ $\mathbf{Bd}_h$ is hypercanonical;*
- *for $h \geq h_m$, $\mathbf{NL}_{m+1} +$ $\mathbf{Bd}_h$ is not strongly complete.*

(ii) Let $m \geq 4$ and let $k_m = depth(\underline{P}_{\sigma_{m,m+1}}) + 1$. Then:

- *for $1 \leq h < k_m$, $\mathbf{NL}_{m+1,m+2} +$ $\mathbf{Bd}_h$ is hypercanonical;*
- *for $h \geq k_m$, $\mathbf{NL}_{m+1,m+2} +$ $\mathbf{Bd}_h$ is not strongly complete.*

Proof:
(i) Let $\underline{K}$ be a model of the logic $\mathbf{NL}_{m+1} +$ $\mathbf{Bd}_h$, with $h < h_m$. Then $\underline{K}$ has at most depth h, hence, noticing that $\underline{P}_{\sigma_m}$ has depth h_m, we have that the frame of $\underline{K}$ is a frame for the logic $\mathbf{NL}_{m+1} +$ $\mathbf{Bd}_h$. If $h \geq h_m$, we can reason as in the proof of Theorem 4.3.6, observing that all the frames of the chain $\mathscr{C}$ have depth h_m, hence they are frames for $\mathbf{Bd}_h$.
(ii) It is proved as (i). □

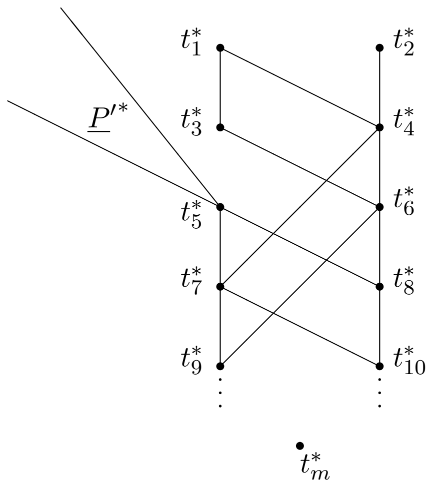

Figure 4.8: The limit frame $\underline{P}^*$

Note that the depth of the frames $\underline{P}_{\sigma_n}$ and $\underline{P}_{\sigma_{m,m+1}}$ can be computed in the following way:

$$\begin{aligned}\text{depth}(\underline{P}_{\sigma_n}) &= (n+1) \text{ DIV } 2\\ \text{depth}(\underline{P}_{\sigma_{m,m+1}}) &= \text{depth}(\underline{P}_{\sigma_{m+1}}) = (m+1) \text{ DIV } 2\end{aligned}$$

where DIV denotes the integer division.

4.3.4 The Anti-Scott logic Ast

It only remains to analyze the logic $\mathbf{NL}_7 = \mathbf{Ast}$ (not included in **St**) which has a peculiar behaviour.

4.3.8 Theorem *The logic* **Ast** *is not strongly complete.*

Proof: Let us consider the chain $\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$ of finite frames $\underline{P}_n = \langle P_n, \leq_n, a_1\rangle$ for **Ast** such that, for every $n \geq 1$, the following holds[3]:

- $P_n = \{a_1, \ldots, a_{n+1}, \alpha, e, b_1, \ldots, b_n, \beta, g, d_1, \ldots, d_n, \delta\}$;
- the ordering relation is defined as in Figure 4.11.

The p-morhpism f_n from $\underline{P}_{n+1}$ onto $\underline{P}_n$ is defined as follows:

- $f_n(a_{n+2}) = \alpha$;
- $f_n(b_{n+1}) = \beta$;

[3]The chain $\mathscr{C}$ is the same as the one in [15].

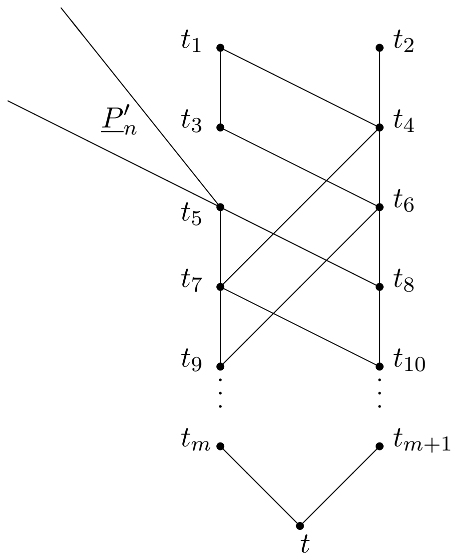

Figure 4.9: The frame $\underline{P}_n$ for $\mathbf{NL}_{m+1,m+2}$

- $f_n(d_{n+1}) = \delta$;
- $f_n(\gamma) = \gamma$ for all the other points γ.

The limit model $\underline{P}^*$ contains the stable points $a_n^* = \langle \alpha, \ldots, \alpha, a_n, a_n, \ldots \rangle$, $b_n^* = \langle \beta, \ldots, \beta, b_n, b_n, \ldots \rangle$, $d_n^* = \langle \delta, \ldots, \delta, d_n, d_n, \ldots \rangle$, $e^* = \langle e, e, \ldots \rangle$, $g^* = \langle g, g, \ldots \rangle$ and the non-stable points $\alpha^* = \langle \alpha, \alpha, \ldots \rangle$, $\beta^* = \langle \beta, \beta, \ldots \rangle$, $\delta^* = \langle \delta, \delta, \ldots \rangle$; the ordering relation is described by Figure 4.12. Let us define a p-morphism g from $\underline{P}^*$ onto $\underline{P}_{\sigma_6}$ as follows:

- $g(\beta^*) = g(g^*) = g(\delta^*) = \sigma_1$;
- $g(d_n^*) = \sigma_2$, for every $n \geq 1$;
- $g(\alpha^*) = g(e^*) = \sigma_3$;
- $g(b_n^*) = \sigma_4$, for every $n \geq 1$;
- $g(a_n^*) = \sigma_6$, for every $n \geq 1$.

By definition of g, $\underline{P}_{\sigma_6}$ is a stable reduction of $\underline{P}^*$; thus, by the Strong Completeness Criterion, **Ast** is not strongly complete. □

We observe that the chain used to disprove the canonicity of **Ast** contains frames of increasing depth, so that the limit has infinite depth. We may wonder whether we can use chains of frames of bounded depth, as in the case of the other non canonical logics in one variable. The answer is negative since, if we fix an upper bound on the depth of the frames, we obtain canonical logics. This fact is not surprising since,

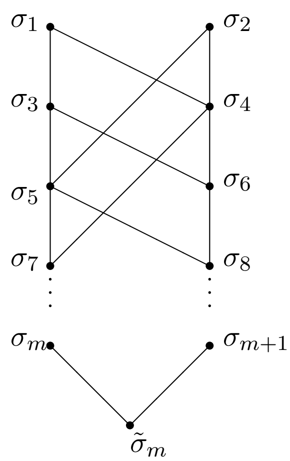

Figure 4.10: The frame $\underline{P}_{\tilde{\sigma}_m}$

using suitable filtration techniques (for instance, the ones explained in [8]), one can prove that the logic **Ast** + **Bd**$_h$ is characterized by the class of frames for **Ast** with depth at most h and, as seen in Proposition 2.5.4, such a class is first-order definable; thus, by Van-Benthem Theorem (Theorem 1.7.3), the canonicity of such a logic follows.

Nevertheless, here we give a direct proof, which enables us to get a more refined classification.

4.3.9 Theorem *The logics* **Ast** + **Bd**$_h$*, for* $h \leq 3$*, are hypercanonical.*

Proof: We only consider the non trivial case of the logic $L =$ **Ast** + **Bd**$_3$. Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a model of the logic L; we show that the frame $\underline{P} = \langle P, \leq \rangle$ is a frame for L. We immediately have that $\underline{P}$ is a frame for **Bd**$_3$; let us suppose that $\underline{P}$ is not a frame for **Ast**. Then there are some points $\alpha, \beta, \gamma, \varphi_1, \varphi_2, \varphi_3$ in $\underline{P}$ such that:

- β and γ are two distinct immediate successors of α;
- φ_1, φ_2 and φ_3 are final points of $\underline{P}$ such that $\varphi_2 \neq \varphi_1$ and $\varphi_2 \neq \varphi_3$;
- $\beta < \varphi_1$, $\gamma < \varphi_2$, $\gamma < \varphi_3$ and $\beta \not< \varphi_2$.

Let V be a finite set of propositional variables such that the points $\alpha, \beta, \gamma, \varphi_1, \varphi_2, \varphi_3$ are pairwise V-separated, with the only exception that $\Gamma^V_{\underline{K}}(\varphi_1)$ may coincide with $\Gamma^V_{\underline{K}}(\varphi_3)$; in particular, we can assume that there is a V-formula A such that $\beta \Vdash A$ and $\varphi_2 \nVdash A$. Let $\underline{K}_V$ be the quotient model of $\underline{K}$ with respect to the V-formulas and, for each $\delta \in P$, let us denote with δ_V the class to which δ belongs (see also

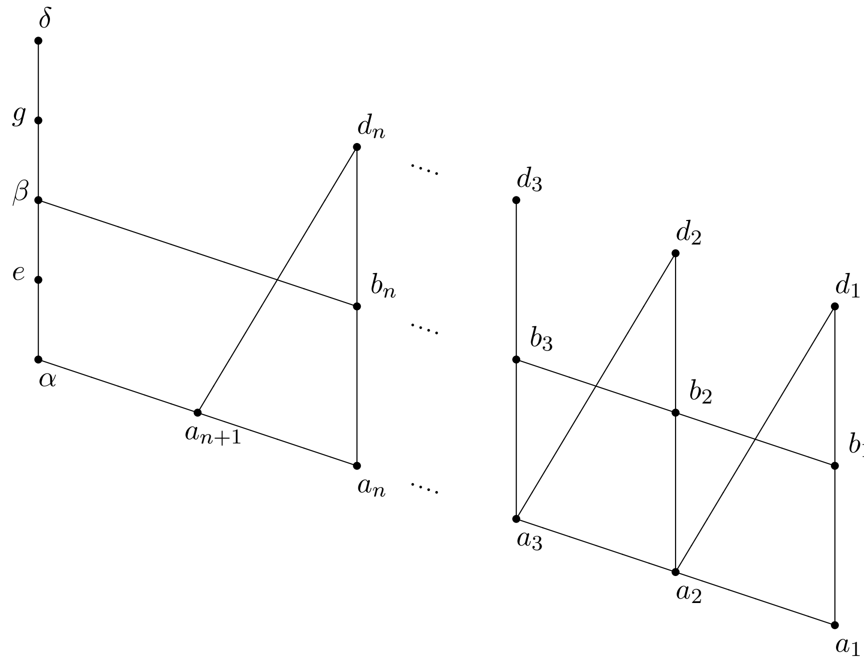

Figure 4.11: The frame $\underline{P}_n$ for **Ast**

Section 1.8). We note that, in $\underline{K}_V$, $\alpha_V < \beta_V < \varphi_{1V}$, $\alpha_V < \gamma_V$, $\gamma_V < \varphi_{2V}$ and $\gamma_V < \varphi_{3V}$; it follows that φ_{1V}, φ_{2V} and φ_{3V} are final, β_V and γ_V have depth 2 and α_V has depth 3, thus β_V and γ_V are immediate successors of α_V. Moreover, since $\underline{K}_V$ is a V-separable finite model of **Ast**, the frame of $\underline{K}_V$ is a finite frame for **Ast**. By Proposition 2.5.4, since γ_V sees more than one final point, β_V and γ_V (which are not final) must see the same final points, hence $\beta_V < \varphi_{2V}$. This gives rise to a contradiction, since $\beta \Vdash A$ and $\varphi_2 \nVdash A$ in $\underline{K}$, which implies that $\beta_V \Vdash A$ and $\varphi_{2V} \nVdash A$ in $\underline{K}_V$. We can conclude that $\underline{P}$ is also a frame for **Ast**. □

If we now try to repeat the reasoning for the logics **Ast** + **Bd**$_h$, with $h \geq 4$, we encounter some difficulties. Indeed, a key point of the proof is that β_V is an immediate successor of α_V in $\underline{K}_V$. Nevertheless, it is not in general true that, if β is an immediate successor of α in $\underline{K}$, then, for some finite set V, β_V is an immediate successor of α_V in $\underline{K}_V$, even if $\underline{K}$ is full. Thus, to overcome the problem, more effort is required and the proof is rather involved.

4.3.10 Theorem *The logics* **Ast** + **Bd**$_h$, *for* $h \geq 4$, *are canonical.*

Proof: Let us suppose that, for some $h \geq 4$, the logic $L =$ **Ast** + **Bd**$_h$ is not canonical. Then there is a full model $\underline{K} = \langle P, \leq, \Vdash \rangle$ of L such that $\underline{P} = \langle P, \leq \rangle$ is not a frame for L. As in the proof of Theorem 4.3.9, since evidently $\underline{P}$ is a frame for **Bd**$_h$, we can assume that there are $\alpha, \beta, \gamma, \varphi_1, \varphi_2, \varphi_3$ in $\underline{P}$ such that:

- β and γ are two distinct immediate successors of α;

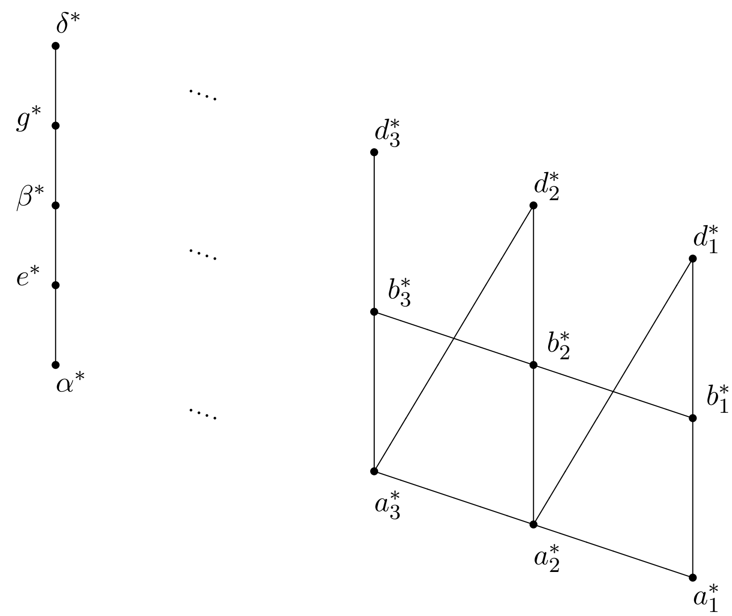

Figure 4.12: The limit frame $\underline{P}^*$

- φ_1, φ_2 and φ_3 are final points of $\underline{P}$ such that $\varphi_2 \neq \varphi_1$ and $\varphi_2 \neq \varphi_3$;
- $\beta < \varphi_1$, $\gamma < \varphi_2$, $\gamma < \varphi_3$ and $\beta \not< \varphi_2$.

Let us consider an increasing sequence of finite sets of propositional variables V_n, for $n \geq 1$, whose union is the set of all the variables; let $\underline{K}_n = \langle P_n \leq_n, \Vdash_n \rangle$ be the quotient model of $\underline{K}$ with respect to the V_n-formulas and let f_n be the map which associates, with each point α in $\underline{K}_{n+1}$, the (unique) point of $\underline{K}_n$ in the same V_n-equivalence class. Since each $\underline{K}_n$ is finite, we have that $\mathscr{C}_K = \{\underline{K}_n, V_n, f_n\}_{n\geq 1}$ is a chain having limit $\underline{K}$, with projections $\{h_n\}_{n\geq 1}$ defined in an obvious way. By the well separability of $\underline{K}$ and by the fact that α has finite depth, we can assume that there is $\overline{n} \geq 1$ such that, for every $j \geq \overline{n}$, the following properties hold:

(P1) for any two distinct points δ_1, δ_2 in the set $\{\alpha, \beta, \gamma, \varphi_1, \varphi_2, \varphi_3\}$, $h_j(\delta_1) \neq h_j(\delta_2)$, with the only exception that $h_j(\varphi_1)$ may coincide with $h_j(\varphi_3)$.

(P2) It is not true that $h_j(\beta) <_j h_j(\varphi_2)$.

(P3) The depth of $h_{j+1}(\alpha)$ in $\underline{K}_{j+1}$ does not exceed the depth of $h_j(\alpha)$ in $\underline{K}_j$.

We observe that, for each $n \geq 1$, $\underline{P}_n$ is a finite frame for **Ast**; moreover, for $j \geq \overline{n}$, it is not the case that all the immediate successors of $h_j(\alpha)$ see only one final point of $\underline{K}_j$ (for instance, if δ is an immediate successor of $h_j(\alpha)$ such that $\delta \leq_j h_j(\gamma)$, then δ sees at least the two distinct final points $h_j(\varphi_2)$ and $h_j(\varphi_3)$); thus, all the non-final immediate successors of $h_j(\alpha)$ see the same final points. In particular:

(P4) for every $j \geq \overline{n}$, for every non-final immediate successor δ of $h_j(\alpha)$ in $\underline{K}_j$, it holds that $\delta <_j h_j(\varphi_2)$.

Now we show that:

(P5) There is $\overline{m} \geq \overline{n}$ such that $h_{\overline{m}}(\beta)$ is an immediate successor of $h_{\overline{m}}(\alpha)$ in $\underline{K}_{\overline{m}}$.

Suppose that (P5) does not hold; then, for every $j \geq \overline{n}$, the set

$$\mathscr{D}_j = \{\delta \in P_j : h_j(\alpha) <_j \delta <_j h_j(\beta) \quad \text{and} \quad \delta <_j h_j(\varphi_2)\}$$

is nonempty. Moreover, it holds that:

(P6) For every $j \geq \overline{n}$ and every $\delta \in \mathscr{D}_{j+1}$, $f_j(\delta) \in \mathscr{D}_j$.

Indeed, since $h_j = f_j \circ h_{j+1}$, we immediately have that $h_j(\alpha) \leq_j f_j(\delta) \leq_j h_j(\beta)$ and $f_j(\delta) <_j h_j(\varphi_2)$. By (P2), it follows that $f_j(\delta) \neq h_j(\beta)$; moreover, it is not true that $f_j(\delta) = h_j(\alpha)$, otherwise, by definition of p-morphism, $h_{j+1}(\alpha)$ would have depth greater than the one of $h_j(\alpha)$, in contradiction with (P3). Thus $f_j(\delta) \in \mathscr{D}_j$ and (P6) holds. By (P6) an by the fact that each $\mathscr{D}_j$ is finite, we can choose an infinite sequence of points $\delta_{\overline{n}} \in \mathscr{D}_{\overline{n}}$, $\delta_{\overline{n}+1} \in \mathscr{D}_{\overline{n}+1}, \ldots$ such that:

$$(*) \quad \delta_{\overline{n}} = f_{\overline{n}}(\delta_{\overline{n}+1}), \quad \delta_{\overline{n}+1} = f_{\overline{n}+1}(\delta_{\overline{n}+2}), \quad \cdots$$

As a matter of fact, we can see the elements of the sets $\mathscr{D}_j$, for every $j \geq \overline{n}$, as the nodes of a tree T, where δ_{j+1} is an immediate successor of δ_j if and only if $\delta_j = f_j(\delta_{j+1})$ (we also have to add a root τ having, as immediate successors, all the elements of $\mathscr{D}_{\overline{n}}$). Since T has infinitely many nodes and each node of T has finitely many immediate successors, by König Lemma (see for instance [21]) T has an infinite branch; clearly the points $\delta_{\overline{n}}, \delta_{\overline{n}+1}, \ldots$ of this branch satisfy (*). Such a sequence generates a point δ^* of the limit $\underline{K}$ of $\mathscr{C}_K$ such that $\alpha < \delta^* < \beta$, against the fact that β is an immediate successor of α in $\underline{K}$. Thus (P5) is proved. By (P5) and (P4) we get that $h_{\overline{m}}(\beta) <_{\overline{m}} h_{\overline{m}}(\varphi_2)$, in contradiction with (P2). This means that the initial hypothesis is false, hence L is canonical. $\square$

We conclude by showing that the logics **Ast** + **Bd**$_h$, for every $h \geq 4$, are not extensively canonical. Let L be any logic of such a family and let us take the chain $\mathscr{C} = \{\underline{P}_n, f_n\}_{n \geq 1}$ defined as follows (see also Figure 4.13).

- For every $n \geq 1$, $\underline{P}_n = \langle P_n, \leq_n, r\rangle$ is the frame such that:
  - $P_n = \{r, a_1, \ldots, a_n, \alpha, b_1, \ldots, b_n, \beta, g_1, \ldots, g_n, \gamma\}$.
  - The immediate successors of the root r are $a_1, \ldots, a_n, \alpha$.
  - For every $1 \leq k \leq n$, the immediate successors of a_k are $g_1, \ldots, g_{k-1}$ (if $k \neq 1$) and b_k.
  - The immediate successors of α are $g_1, \ldots, g_n$ and β.

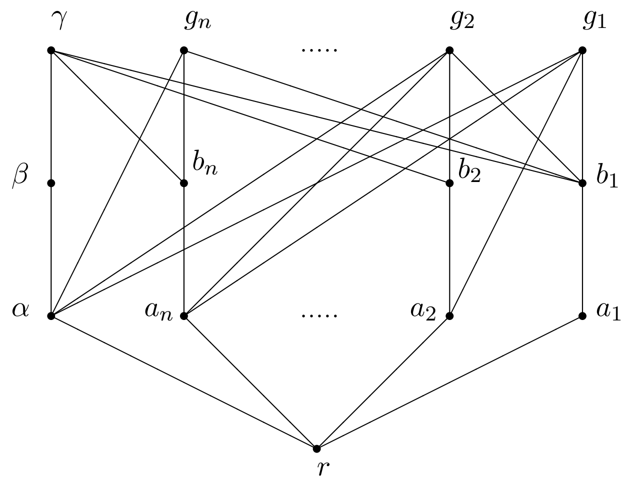

Figure 4.13: The frame $\underline{P}_n$ for **Ast** + **Bd**$_h$

- For every $1 \leq k \leq n$, the immediate successors of b_k are $g_k, \ldots, g_n, \gamma$.
- The only immediate successor of β is γ.
- $g_1, \ldots, g_n, \gamma$ are final points of $\underline{P}_n$.

- The p-morphism f_n from $\underline{P}_{n+1}$ onto $\underline{P}_n$ is defined as follows.
  - $f_n(a_{n+1}) = \alpha$.
  - $f_n(b_{n+1}) = \beta$.
  - $f_n(g_{n+1}) = \gamma$.
  - $f_n(\delta) = \delta$ for all the other points δ.

It is easy to check that $\underline{P}_n$ is a frame for **Ast** (note that the immediate successors of r see all the final points of $\underline{P}_n$); moreover, since depth$(\underline{P}_n) = 4$, we can state that $\underline{P}_n$ is a frame for L. Let $\underline{P} = \langle P, \leq, r \rangle$ be the infinite frame defined as follows (see Figure 4.14).

- $P = \{r, a_1, a_2, \ldots, b_1, b_2 \ldots, \beta, g_1, g_2, \ldots, \gamma\}$.
- The immediate successors of the root r are the points a_n, for every $n \geq 1$, and β.
- For every $n \geq 1$, the immediate successors of a_n are $g_1, \ldots, g_{n-1}$ (if $n \neq 1$) and b_n.
- For every $n \geq 1$, the immediate successors of b_n are the points g_k, for every $k \geq n$, and γ.

- The only immediate successor of β is γ.
- The points g_n, for every $n \geq 1$, and γ are final.

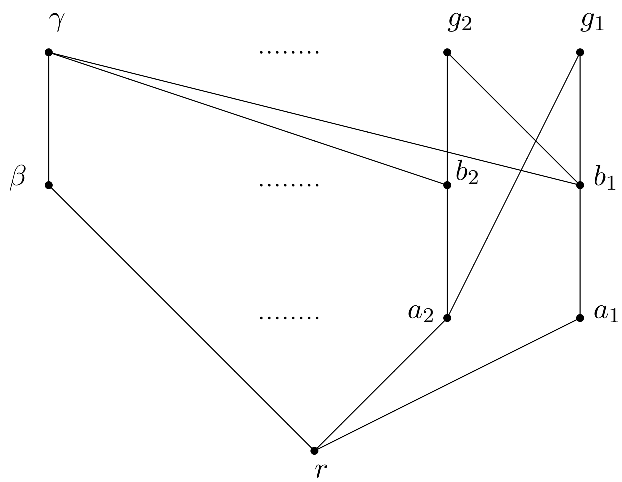

Figure 4.14: The well separable weak limit $\underline{P}$

We claim that $\underline{P}$ is a well separable weak limit of $\mathscr{C}$ having projections $\{h_n\}_{n\geq 1}$ defined as follows:

- $h_n(a_k) = \alpha$ for every $k \geq n+1$.
- $h_n(b_k) = \beta$ for every $k \geq n+1$.
- $h_n(g_k) = \gamma$ for every $k \geq n+1$.
- $h_n(\delta) = \delta$ for all the other points δ.

Since evidently $\underline{P}$ is not a frame for **Ast**, by the Extensive Canonicity Criterion we can deduce that L is not extensively canonical. Clearly $\underline{P}$ is not isomorphic to the limit of $\mathscr{C}$ since, taking the points $\alpha \in P_1, \alpha \in P_2, \ldots$ we have that $h_n(\alpha) = \alpha$ for every $n \geq 1$, but there is not any point δ of $\underline{P}$ such that $h_n(\delta) = \alpha$ for all $n \geq 1$. To obtain the limit, we have to insert in $\underline{P}$ a point α such that α is an immediate successor of r and β, $g_1, g_2, \ldots$ are all the immediate successors of α, as in Figure 4.15. One can also check that such a frame is actually a frame for L, according to the fact that L is canonical and to the Canonicity Criterion. This example is particularly interesting, since it shows that extensive ω-canonicity does not imply extensive canonicity; indeed, all the logics **Ast** + $\mathbf{Bd}_h$ are extensively ω-canonical, due to the fact that, for every finite V, the well V-separable models of such logics are finite (see Proposition 1.9.8).

To sum up:

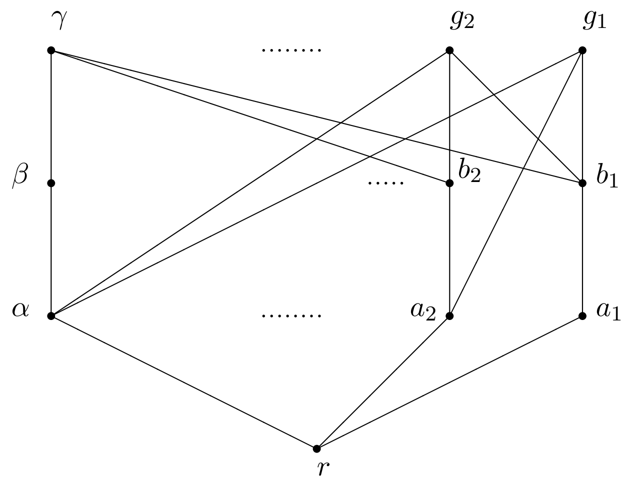

Figure 4.15: The limit $\underline{P}^*$

4.3.11 Corollary

(i) The logics $\mathbf{Ast}+\ \mathbf{Bd}_h$*, for* $h \leq 3$*, are hypercanonical.*

(ii) The logics $\mathbf{Ast}+\ \mathbf{Bd}_h$*, for* $h \geq 4$*, are canonical, extensively* ω*-canonical but not extensively canonical.*

□

4.4 Non extensive canonicity of Medvedev logic

As a minor application of our criteria, we show that **MV** is not extensively canonical; to obtain stronger results, we need a more careful knowledge of the semantics of **MV**. Let, for each $n \geq 1$, $X_n = \{1, \ldots, n\}$ and let us consider the chain of frames $\mathscr{C} = \{\underline{P}_n, f_n\}_{n \geq 1}$ defined as follows:

- $\underline{P}_n$ is the MV-frame determined by X_n (see Figure 4.16);
- for every $Y \in P_{n+1}$, $f_n(Y) = \{s_n(y) : y \in Y\}$

where $s_n(y) = y$ if $y \leq n$, $s_n(y) = n$ otherwise. Note that, due to the special definition of MV-frame, the possible p-morphisms from $\underline{P}_{n+1}$ onto $\underline{P}_n$ are trivial permutations of f_n. Let $X^+ = \mathbb{N} \cup \{\omega\}$ (where $\mathbb{N}$ is the set of natural numbers) and let $\underline{P} = \langle P, \leq \rangle$ be the frame defined as follows:

- $P = \{X_n : n \geq 1\} \cup \{\{\omega\}\} \cup \{X^+\}$.
- For every $Y, Z \in P$, $Y \leq Z$ iff $Z \subseteq Y$.

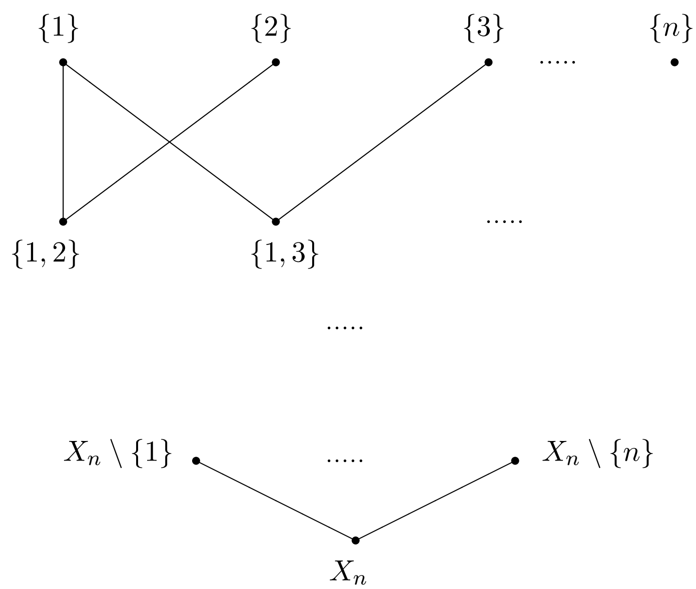

Figure 4.16: The MV-frame $\underline{P}_n$

Note that X^+ is the root of $\underline{P}$ and $\{\omega\}$ is the only immediate successor of X^+ (see Figure 4.17). We claim that $\underline{P}$ is a well separable weak limit of $\mathscr{C}$ with projections $\{h_n\}_{n\geq 1}$ defined as follows:

- for every $Y \in P$, $h_n(Y) = \{s_n^+(y) : y \in Y\}$

where $s_n^+(y) = s_n(y)$ if $y \in \mathbb{N}$, and $s_n^+(\omega) = n$. Clearly, $\underline{P}$ is not a frame for **MV**; as a matter of fact, $\underline{P}$ is neither a frame for **St** nor a frame for **KP** (we recall that both these logics are contained in **MV**). By the Extensive Canonicity Criterion, we can conclude that:

4.4.1 Theorem *The Medvedev logic* **MV** *is not extensively canonical.* □

Note that, to obtain the limit $\underline{P}^*$ of $\mathscr{C}$, we have to add all the sets of the kind $X_n \cup \{\omega\}$; we can apply Proposition 1.3.1 and state that $\underline{P}^*$ is a frame for **MV**. Thus, we cannot use this kind of chains in order to disprove the canonicity of **MV**, and the question, as far as we know, remains open.

4.5 Non canonicity of the logic of rhombuses

In this section, we prove that the logic of rhombuses is not canonical. Let T^+ be linearly ordered set $\{1, 2, \ldots, n, \ldots, \omega\}$. We define a chain of frames $\mathscr{C} = \{\underline{P}_n, f_n\}_{n\geq 1}$, where $\underline{P}_n$ is the RH-frame defined on the linear ordering $T_n = \{1, 2, \ldots, n, \omega\}$ (see Figure 4.18); the p-morphism f_n from $\underline{P}_{n+1}$ onto $\underline{P}_n$ is defined in an obvious way. More precisely, let g_n be the map on the integers defined as follows:

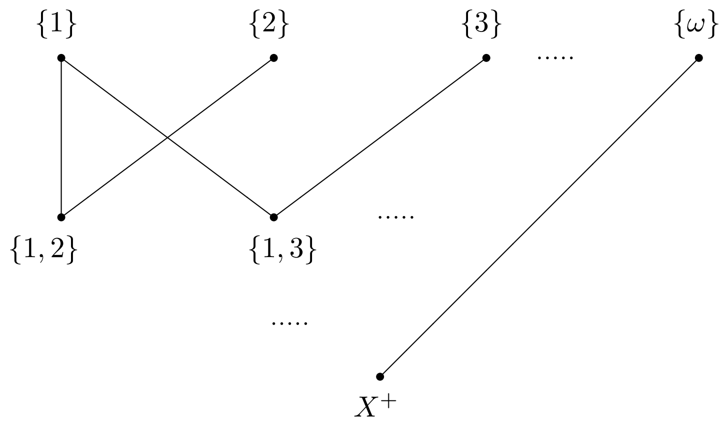

Figure 4.17: The well separable weak limit $\underline{P}$

- $g_n(k) = k$ if $k \leq n$;

- $g_n(k) = \omega$ if $k > n$.

Then:

- $f_n([k,l]) = [g_n(k), g_n(l)]$.

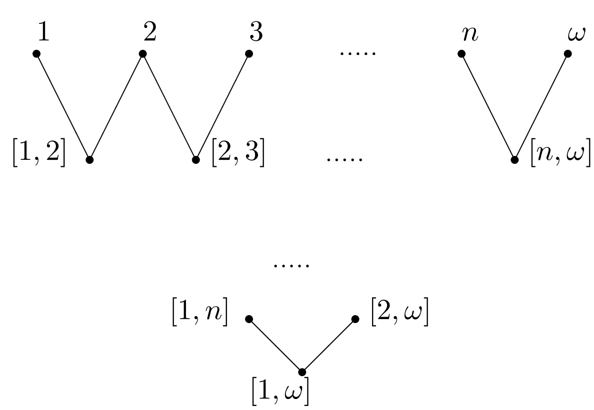

Figure 4.18: The RH-frame $\underline{P}_n$

The limit $\underline{P}^* = \langle P^*, \leq^* \rangle$ of $\mathscr{C}$ is isomorphic to the frame $\underline{P} = \langle P, \leq \rangle$ defined as follows (see Figure 4.19):

- $P = \{[a,b] : a, b \in T^+ \text{ and } a \leq b\}$;

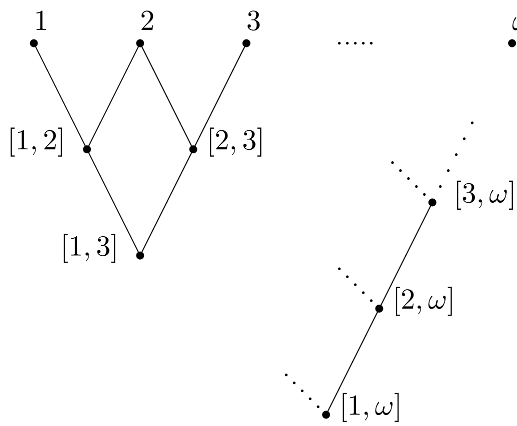

Figure 4.19: The limit $\underline{P}$

- $[a, b] \leq [c, d]$ iff $[c, d] \subseteq [a, b]$.

We point out that $[1, \omega]$ is the root of $\underline{P}$ and the stable points of $\underline{P}$ are the ones of the kind $[a, b]$ with $b < \omega$. It is easy to see that $\underline{P}$ is not a frame for **St**, indeed we can define a p-morphism g from $\underline{P}$ onto $\underline{P}_{\sigma_5}$ as follows:

- $g(k) = \sigma_1$ if $k < \omega$ (where k, as usual, denotes the interval $[k, k]$).

- $g(\omega) = \sigma_2$.

- $g([k, l]) = \sigma_3$ if $k \neq l$ and $l \neq \omega$.

- $g([k, \omega]) = \sigma_5$ if $k \neq \omega$.

We stress that g is not a V-stable reduction and, with this kind of chains, it seems difficult to find a counterexample which allows us to apply the Strong Completeness Criterion. In the present case, for instance, we can even find a weak limit $\underline{P}'$ of $\mathscr{C}$ which is a frame for **RH**. As a matter of fact, let $\underline{P}'$ be the subframe of $\underline{P}$ obtained by considering the point $[1, \omega]$ and all the stable points of $\underline{P}$ (see Figure 4.20). Then, it is easy to check that $\underline{P}'$ is a (well separable) weak limit of $\mathscr{C}$. Moreover, it is immediate to see that $\underline{P}'$ has the filter property, therefore, by Proposition 1.3.1, $\underline{P}'$ is a frame for **RH**. Thus, as a consequence of Proposition 4.2.5, all the stable reductions of the limit $\underline{P}$ of $\mathscr{C}$ are frames for **RH**. Also in this case, to strengthen the result, we need more knowledge about the semantics of **RH**.

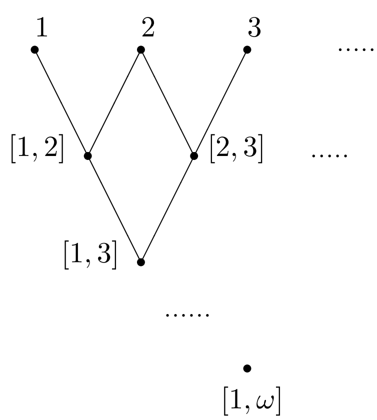

Figure 4.20: The infinite frame $\underline{P}'$ for **RH**

Chapter 5

Analysis of ω-Canonicity

This chapter presents the most original results of the thesis. We give a criterion for the strong ω-completeness and we apply it in order to prove the following remarkable result, which considerably improves the one stated at the beginning of previous chapter:

- All the intermediate logics axiomatized by axioms in one variable, except eight of them, are not strongly ω-complete.

This completes the classification of the logics in one variable.

To state such results, we could adapt the arguments of the previous chapter to the context of strong ω-completeness (a similar approach is followed in [15]). Indeed, we can reformulate the notions explained in the previous chapter and relativize them to the V-formulas. More precisely: we can define the notion of V-chain, the various notions of V-limit, and so on; then, the ω-canonicity and strong ω-completeness criteria become formally similar to the canonicity and strong completeness criteria respectively. On the other hand, in the definition of V-chain, the map f_n from $\underline{P}_{n+1}$ onto $\underline{P}_n$ cannot be simply a p-morphism, but has to satisfy stronger properties; this is due to the fact that it is not sufficient that f_n preserves the V-formulas, but we need the stronger requirement that f_n preserves the V-formulas "up to the implicational complexity n". The drawback in this approach is that the construction of counterexamples, by means of V-chains, becomes rather involved; we are not even sure that, following this method, we are able to build the complex countermodels used in this chapter (incidentally, we point out that this technique is used in [15] only to disprove the strong ω-completeness of the logics $\mathbf{T}_n$).

Our approach is completely different and more loose: instead of building a V-separable and V-full model $\underline{K}$ by means of a V-chain, we directly define the countermodel $\underline{K}$ (which is in general a "big" model) and successively we take care to prove the V-separability and the V-fullness of $\underline{K}$. At this aim, we classify the points of $\underline{K}$ using the notion of V-grade, which allows us to check the V-fullness of $\underline{K}$. Roughly speaking, suppose that $\underline{K}$ is a V-full model and suppose that $\underline{K}$ coincides with the limit of a V-chain of models $\mathscr{C}_K^V$; then the points of finite V-grade represent the "stable" points of $\underline{K}$, while the points of infinite V-grade correspond to the "instable" points of $\underline{K}$. As in the framework of strong completeness, this distinction is crucial for the analysis of strong ω-completeness. The material of this chapter is almost entirely reported in [10].

5.1 Some conditions about strong ω-completeness

We now tackle the problem of ω-canonicity and strong ω-completeness. As a key tool in describing models, we introduce the relations $\preccurlyeq_n^V$ and $\sim_n^V$.

5.1.1 Definition *Let V be a finite set of propositional variables; let $\underline{K} = \langle P, \leq, \Vdash \rangle$ and $\underline{K}' = \langle P', \leq', \Vdash' \rangle$ be (non necessarily different) Kripke models. Then the relations $\preccurlyeq_n^V$ and $\sim_n^V$ between points of $\underline{K}$ and $\underline{K}'$ are defined, inductively on $n \geq 0$, by the following conditions. Let $\alpha \in P$ and $\alpha' \in P'$; then:*

- *$\alpha \preccurlyeq_0^V \alpha'$ iff, for all $p \in V$, $\alpha \Vdash p$ implies $\alpha' \Vdash' p$.*
- *$\alpha \sim_0^V \alpha'$ iff $\alpha \preccurlyeq_0^V \alpha'$ and $\alpha' \preccurlyeq_0^V \alpha$.*
- *$\alpha \preccurlyeq_{n+1}^V \alpha'$ iff, for all $\beta' \in P'$ such that $\alpha' \leq' \beta'$, there is $\beta \in P$ such that $\alpha \leq \beta$ and $\beta \sim_n^V \beta'$.*
- *$\alpha \sim_{n+1}^V \alpha'$ iff $\alpha \preccurlyeq_{n+1}^V \alpha'$ and $\alpha' \preccurlyeq_{n+1}^V \alpha$.*

□

It is easy to see that $\alpha \preccurlyeq_n^V \alpha'$ implies $\alpha \preccurlyeq_k^V \alpha'$ for every $k \leq n$; hence, $\alpha \sim_n^V \alpha'$ implies $\alpha \sim_k^V \alpha'$ for every $k \leq n$. We remark that, for every $n \geq 0$, $\sim_n^V$ is an equivalence relation having *finitely* many equivalence classes.

5.1.2 Definition *Let A be any formula. The* implicational complexity *$Ic(A)$ of A is defined, by induction on the (structural) complexity of A, by the following conditions.*

- *$Ic(A) = 0$ if A is atomic.*
- *$Ic(A) = max\{Ic(B), Ic(C)\}$ if either $A = B \wedge C$ or $A = B \vee C$.*
- *$Ic(B \to C) = max\{Ic(B), Ic(C)\} + 1$.*
- *$Ic(\neg A) = Ic(A) + 1$.*

□

For a model $\underline{K} = \langle P, \leq, \Vdash \rangle$ and a finite set V, $\Gamma^{V,n}_{\underline{K}}(\alpha)$ (or simply $\Gamma^{V,n}(\alpha)$ if the context is clear) denotes the set of V-formulas of implicational complexity $r \leq n$ forced in α.
A remarkable point is that the equivalence classes $[\alpha]_{\sim^V_n}$ of the relations $\sim^V_n$ are characterized by formulas H^n_α of implicational complexity at most n (see also [16]); we give a proof of this fact in next lemma. To simplify the statement of the lemma, we assume to have $\top$ and $\bot$, intuitionistically equivalent to $p \to p$ and $\neg(p \to p)$ respectively, as primitive atomic symbols of the language; moreover, we say that two formulas A and B essentially coincides if B is obtained from A by trivial permutations between the formulas C_1 and C_2 in subformulas of A of the kind $C_1 \wedge C_2$ and $C_1 \vee C_2$.

5.1.3 Lemma *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a Kripke model and let V be a finite set of propositional variables. For every $n \geq 0$ and every $\alpha \in P$, there is a V-formula H^n_α such that:*

(1) $Ic(H^n_\alpha) \leq n$;

(2) For every $\underline{K}' = \langle P', \leq', \Vdash' \rangle$ and every $\alpha' \in P'$, if $\alpha' \sim^V_n \alpha$ then $H^n_{\alpha'}$ essentially coincides with H^n_α;

(3) For every $\underline{K}' = \langle P', \leq', \Vdash' \rangle$ and every $\alpha' \in P'$, $\alpha' \Vdash' H^n_\alpha$ iff $\alpha \preccurlyeq^V_n \alpha'$.

Proof: By induction on n. Suppose $n = 0$. If α does not force any $p \in V$, we can set $H^0_\alpha = \top$ and the lemma is satisfied (indeed, $\alpha \preccurlyeq^V_0 \alpha'$ for all α'); otherwise we take:

$$H^0_\alpha = \bigwedge_{p \in V : \alpha \Vdash p} p.$$

Suppose now, by the induction hypothesis, that the lemma holds for n. Let, for any $\alpha \in P$:

$$\begin{aligned} \mathscr{S}(\alpha) &= \{\beta : \beta \text{ is any point of any Kripke model and,} \\ &\qquad \text{for every } \delta \in P,\ \alpha \leq \delta \text{ implies } \beta \not\sim^V_n \delta\}; \\ \mathscr{S}'(\alpha) &= \{\beta : \beta \text{ is any point of any Kripke model and } \beta \not\preccurlyeq^V_n \alpha\}. \end{aligned}$$

If $\mathscr{S}(\alpha) = \emptyset$, then $\alpha \preccurlyeq^V_{n+1} \alpha'$ for every α' and, as before, we can set $H^{n+1}_\alpha = \top$. Otherwise, we define:

$$H^{n+1}_\alpha = \bigwedge_{\beta \in \mathscr{S}(\alpha)} (H^n_\beta \longrightarrow \bigvee_{\delta \in \mathscr{S}'(\beta)} H^n_\delta \vee \bot).$$

We remark that, in the above formula, we assume to take only one characteristic formula in correspondence of $\sim^V_n$ -equivalent points in $\mathscr{S}(\alpha)$ and in $\mathscr{S}'(\beta)$ respectively. By the fact that there are finitely many non equivalent points with respect to $\sim^V_n$ and by the induction hypothesis, it immediately follows that H^{n+1}_α is a well defined

V-formula which satisfies (1) and (2). In order to prove (3), let $\underline{K}' = \langle P', \leq', \Vdash' \rangle$ be any Kripke model and let $\alpha' \in P'$. Suppose that $\alpha \preccurlyeq^V_{n+1} \alpha'$; we prove that $\alpha' \Vdash' H^{n+1}_\alpha$. Take any $\delta' \in P'$ such that:
- $\alpha' \leq' \delta'$;
- $\delta' \Vdash' H^n_\beta$ for some $\beta \in \mathscr{S}(\alpha)$.

By the induction hypothesis, $\beta \preccurlyeq^V_n \delta'$. Suppose now that $\delta' \preccurlyeq^V_n \beta$. Then $\delta' \sim^V_n \beta$ and, since $\alpha \preccurlyeq^V_{n+1} \alpha'$ and $\alpha' \leq' \delta'$, there is $\delta \in P$ such that $\alpha \leq \delta$ and $\delta \sim^V_n \delta'$. This implies that $\delta \sim^V_n \beta$, against the fact that $\beta \in \mathscr{S}(\alpha)$; thus $\delta' \not\preccurlyeq^V_n \beta$, that is $\delta' \in \mathscr{S}'(\beta)$. Since $\delta' \Vdash' H^n_{\delta'}$ (by the induction hypothesis, being $\delta' \preccurlyeq^V_n \delta'$), we get:

$$\alpha' \Vdash' H^n_\beta \longrightarrow \bigvee_{\delta \in \mathscr{S}'(\beta)} H^n_\delta \vee \bot.$$

By the generality of δ' and β, it follows that $\alpha' \Vdash' H^{n+1}_\alpha$.

Suppose now that $\alpha \not\preccurlyeq^V_{n+1} \alpha'$. There is $\beta' \in P'$ such that:
- $\alpha' \leq' \beta'$;
- for every $\delta \in P$ such that $\alpha \leq \delta$, $\beta' \not\sim^V_n \delta$.

By the induction hypothesis, we have:
- $\beta' \Vdash' H^n_{\beta'}$;
- $\delta \in \mathscr{S}'(\beta')$ implies $\beta' \nVdash' H^n_\delta$ (in fact, $\delta \not\preccurlyeq^V_n \beta'$).

Note that $\mathscr{S}'(\beta')$ may be empty. Therefore:

$$\alpha' \nVdash' H^n_{\beta'} \longrightarrow \bigvee_{\delta \in \mathscr{S}'(\beta')} H^n_\delta \vee \bot.$$

Since, by the above assumptions, $\beta' \in \mathscr{S}(\alpha)$, we can conclude that $\alpha' \nVdash' H^{n+1}_\alpha$. □

Now we show that the $\sim^V_n$ equivalences preserve the forcing of V-formulas up to the implicational complexity n.

5.1.4 Proposition *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ and $\underline{K}' = \langle P', \leq', \Vdash' \rangle$ be any two Kripke models, and let V be a finite set of propositional variables. Then, for every $\alpha \in P$, $\alpha' \in P'$ and every $n \geq 0$, it holds that:*

(i) $\alpha \preccurlyeq^V_n \alpha'$ *if and only if* $\Gamma^{V,n}_{\underline{K}}(\alpha) \subseteq \Gamma^{V,n}_{\underline{K}'}(\alpha')$.

(ii) $\alpha \sim^V_n \alpha'$ *if and only if* $\Gamma^{V,n}_{\underline{K}}(\alpha) = \Gamma^{V,n}_{\underline{K}'}(\alpha')$.

Proof: (i) Firstly we show that, for every point α of $\underline{K}$ and α' of $\underline{K}'$ and for every V-formula A, the following property holds:

(*) $\alpha \Vdash A$ and $\mathrm{Ic}(A) \leq n$ and $\alpha \preccurlyeq^V_n \alpha'$ implies $\alpha' \Vdash' A$.

We proceed by induction on the structure of A. If A is atomic then, by the fact that $\alpha \preccurlyeq^V_0 \alpha'$, (*) immediately follows. If $A = B \wedge C$ or $A = B \vee C$, (*) follows from the induction hypothesis. Let us suppose that $A = B \to C$ and $\alpha' \nVdash' B \to C$.

Then there is $\beta' \geq' \alpha'$ such that $\beta' \Vdash' B$ and $\beta' \nVdash' C$. Let $\beta \in P$ be such that $\alpha \leq \beta$ and $\beta \sim^V_{n-1} \beta'$, that is $\beta \preccurlyeq^V_{n-1} \beta'$ and $\beta' \preccurlyeq^V_{n-1} \beta$. Since $\mathrm{Ic}(B) \leq n-1$ and $\mathrm{Ic}(C) \leq n-1$, we can apply the induction hypothesis and state that $\beta \Vdash B$ and $\beta \nVdash C$, thus $\alpha \nVdash B \to C$ and (*) is proved. It follows that $\alpha \preccurlyeq^V_n \alpha'$ implies $\Gamma^{V,n}_{\underline{K}}(\alpha) \subseteq \Gamma^{V,n}_{\underline{K}'}(\alpha')$. Conversely, let us suppose that $\alpha \not\preccurlyeq^V_n \alpha'$. By Lemma 5.1.3, it holds that $\alpha \Vdash H^n_\alpha$ and $\alpha' \nVdash H^n_\alpha$, thus $\Gamma^{V,n}_{\underline{K}}(\alpha) \not\subseteq \Gamma^{V,n}_{\underline{K}'}(\alpha')$, and this completes the proof of (i).
(ii) It is an immediate consequence of (i). □

5.1.5 Definition *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a Kripke model, let V be a finite set of propositional variables and let $\alpha \in P$. We say that α has V-*grade r (in $\underline{K}$) *iff r is the minimum $k \geq 0$ such that the following condition holds:*

- *for every $\beta \in P$, $\beta \sim^V_k \alpha$ implies $\beta = \alpha$.*

□

We say that a point α of $\underline{K}$ has *finite* V-grade if it has V-grade $r \geq 0$; otherwise, α has *infinite* V-grade. We remark that if α has V-grade r in $\underline{K}$, then α has V-grade $r' \leq r$ in any generated submodel $\underline{K}'$ of $\underline{K}$ in which α is defined; on the other hand, a point having infinite V-grade in $\underline{K}$ may have finite V-grade in some generated submodel of $\underline{K}$. Note also that a final point of a V-separable model $\underline{K}$ has V-grade 0 or 1; it follows that, for every δ, δ' in $\underline{K}$, $\delta \sim^V_2 \delta'$ implies $\mathrm{Fin}(\delta) = \mathrm{Fin}(\delta')$.

5.1.6 Proposition *Let $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ be a V-separable and V-full Kripke model (V finite) and let $\underline{K}' = \langle P', \leq', \rho', \Vdash' \rangle$ be any Kripke model such that $\Gamma^V_{\underline{K}'}(\rho') = \Gamma^V_{\underline{K}}(\rho)$. Then there is a map $h : P' \to P$ such that:*

(i) $h(\rho') = \rho$;

(ii) $\alpha' \leq' \beta'$ *implies* $h(\alpha') \leq h(\beta')$;

(iii) if $h(\alpha') \leq \beta$ and β has finite V-grade in $\underline{K}$, there is $\beta' \in P'$ s.t. $\alpha' \leq' \beta'$ and $h(\beta') = \beta$.

Proof: Let α' be any point of P'; then $\Gamma^V_{\underline{K}'}(\rho') \subseteq \Gamma^V_{\underline{K}'}(\alpha')$, hence $\Gamma^V_{\underline{K}}(\rho) \subseteq \Gamma^V_{\underline{K}'}(\alpha')$. Since $\underline{K}$ is V-full and V-separable, there is one and only one $\alpha \in P$ such that $\Gamma^V_{\underline{K}}(\alpha) = \Gamma^V_{\underline{K}'}(\alpha')$. So we are allowed to define $h : P' \to P$ as follows:

$$h(\alpha') = \alpha \text{ if and only if } \Gamma^V_{\underline{K}}(\alpha) = \Gamma^V_{\underline{K}'}(\alpha')$$

(note that $\alpha' \sim^V_k h(\alpha')$ for every $k \geq 0$). Obviously $h(\rho') = \rho$; moreover, since $\underline{K}$ is well V-separable, (ii) immediately follows. Suppose now that $h(\alpha') \leq \beta$ and that β has finite V-grade $r \geq 0$ in $\underline{K}$. Since $h(\alpha') \sim^V_{r+1} \alpha'$, there is $\beta' \in P'$ such that $\alpha' \leq' \beta'$ and $\beta \sim^V_r \beta'$. Since $\beta' \sim^V_r h(\beta')$, it follows that $\beta \sim^V_r h(\beta')$; but β has V-grade r in $\underline{K}$, hence $\beta = h(\beta')$ and (iii) is proved as well. □

We point out that h may be non-surjective; indeed, the points of $\underline{K}$ of infinite V-grade may not have any preimage in $\underline{K}'$ (while all the points of finite V-grade have at least one preimage); moreover, if $\underline{K}'$ is not V-separable, h is not injective.
As far as the V-separability of points of finite V-grade is concerned, we can assert:

5.1.7 Lemma *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a Kripke model and let α be a point of $\underline{K}$ of finite V-grade in $\underline{K}$. Then, for every $\beta \in P$, $\Gamma^V_{\underline{K}}(\alpha) = \Gamma^V_{\underline{K}}(\beta)$ implies $\alpha = \beta$.*

Proof: If $\Gamma^V_{\underline{K}}(\alpha) = \Gamma^V_{\underline{K}}(\beta)$, then $\alpha \sim^V_k \beta$ for every $k \geq 0$. Let $r \geq 0$ be the V-grade of α; since $\alpha \sim^V_r \beta$, it holds that $\alpha = \beta$. □

By this lemma, we can give the following condition on V-separability.

5.1.8 Proposition *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a Kripke model and let V be a finite set of propositional variables. $\underline{K}$ is V-separable if and only if, for every $\alpha, \beta \in P$ of infinite V-grade, $\Gamma^V_{\underline{K}}(\alpha) = \Gamma^V_{\underline{K}}(\beta)$ implies $\alpha = \beta$.* □

In order to give a condition for the V-fullness, we introduce the notion of V-sequence.

5.1.9 Definition *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a Kripke model and let V be a finite set of propositional variables. A V-*sequence* $\{\beta_k\}^V_{k \geq 0}$ of $\underline{K}$ is a sequence of points $\beta_k \in P$ such that $\beta_k \sim^V_k \beta_{k+1}$ for every $k \geq 0$.*
We say that $\beta \in P$ is a limit *of the V-sequence $\{\beta_k\}^V_{k \geq 0}$ if and only if $\beta \sim^V_k \beta_k$ for every $k \geq 0$.* □

We stress that the point β_k of a V-sequence can be viewed as an approximation of β (with respect to the forcing of V-formulas) up to the V-formulas of implicational complexity k. We also remark that, if $\underline{K}$ is V-separable, then the limit is unique. The necessary and sufficient condition for the V-fullness is stated in the following theorem.

5.1.10 Proposition *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a Kripke model and let V be a finite set of propositional variables. Then $\underline{K}$ is V-full if and only if, for every cone $\underline{K}_\alpha$ of $\underline{K}$, every V-sequence $\{\beta_k\}^V_{k \geq 0}$ of $\underline{K}_\alpha$ has a limit in $\underline{K}_\alpha$.*

Proof: Suppose that $\underline{K}$ is V-full; let $\underline{K}_\alpha$ be a cone of $\underline{K}$ and let $\{\beta_k\}^V_{k \geq 0}$ be a V-sequence of $\underline{K}_\alpha$. Let us consider the set:

$$\Delta^V = \bigcup_{k \geq 0} \Gamma^{V,k}_{\underline{K}}(\beta_k).$$

Then Δ^V is a V-saturated set and $\Gamma^V_{\underline{K}}(\alpha) \subseteq \Delta^V$; by the V-fullness of $\underline{K}$, there is $\beta \in P$ such that $\alpha \leq \beta$ and $\Gamma^V_{\underline{K}}(\beta) = \Delta^V$. By definition of V-sequence, $\Gamma^{V,k}_{\underline{K}}(\beta_k) = \Gamma^{V,k}_{\underline{K}}(\beta_{k+j})$ for every $k, j \geq 0$, which implies that $\Gamma^{V,k}_{\underline{K}}(\beta) = \Gamma^{V,k}_{\underline{K}}(\beta_k)$ for every $k \geq 0$,

hence $\beta \sim_k^V \beta_k$ for every $k \geq 0$. We can conclude that the point β of $\underline{K}_\alpha$ is a limit of the V-sequence $\{\beta_k\}_{k\geq 0}^V$.
Conversely, let us suppose that $\underline{K}$ satisfies the condition on V-sequences; let α be any point of $\underline{K}$ and let Δ^V be a V-saturated set such that $\Gamma_{\underline{K}}^V(\alpha) \subseteq \Delta^V$. Let $\underline{K}^* = \langle P^*, \leq^*, \rho^*, \Vdash^* \rangle$ be a V-full model such that $\Gamma_{\underline{K}^*}^V(\rho^*) = \Gamma_{\underline{K}}^V(\alpha)$ (for instance, we can take the cone of the V-canonical model for **Int** generated by the V-saturated set $\Gamma_{\underline{K}}^V(\alpha)$); by the V-fullness of $\underline{K}^*$, there is β^* in $\underline{P}^*$ such that $\Gamma_{\underline{K}^*}^V(\beta^*) = \Delta^V$. Since $\Gamma_{\underline{K}^*}^V(\rho^*) = \Gamma_{\underline{K}}^V(\alpha)$, it holds that:

$$\rho^* \sim_k^V \alpha \quad \text{for every } k \geq 0.$$

By the properties of the relations $\sim_k^V$, we can determine a sequence of points β_k of $\underline{P}_\alpha$ such that:

$$\beta^* \sim_k^V \beta_k \quad \text{for every } k \geq 0.$$

It follows that $\beta_{k+1} \sim_k^V \beta_k$ for every $k \geq 0$, therefore $\{\beta_k\}_{k\geq 0}^V$ is a V-sequence of $\underline{K}_\alpha$. By the hypothesis of the proposition, such a V-sequence has a limit β in $\underline{K}_\alpha$; this means that $\alpha \leq \beta$ and, since $\beta^* \sim_k^V \beta$ for every $k \geq 0$, we can conclude that $\Gamma_{\underline{K}^*}^V(\beta^*) = \Gamma_{\underline{K}}^V(\beta)$, hence $\Gamma_{\underline{K}}^V(\beta) = \Delta^V$ and $\underline{K}$ is V-full. □

We remark that the condition of Proposition 5.1.10 is trivially satisfied by the V-sequences $\{\beta_k\}_{k\geq 0}^V$ definitively constant. Indeed, if $\beta_k = \beta_n$ for every $k \geq n$, then β_n is the limit of the V-sequence; thus, in applying the proposition, we can limit ourselves to consider V-sequences which are not definitively constant.

With the tools described so far, we can build V-separable and V-full models (and this will be done in next sections) and we can find "counterexamples" for the ω-canonicity. On the other hand (as in the case of the strong completeness), if our concern is to disprove the strong ω-completeness of some logic L, it is not sufficient to take in account the V-separable and V-full models of a L, V-saturated set Δ^V, but *all* the models of Δ^V. We argue as in the previous chapter and we introduce the notion of V-stable reduction that plays the role of the one of stable reduction.

5.1.11 Definition *Let $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ be a Kripke model and let V be a finite set of propositional variables. We say that $\underline{P}' = \langle P', \leq', \rho' \rangle$ is a* V-stable reduction *of $\underline{P} = \langle P, \leq, \rho \rangle$ if and only if there is a p-morphism g from $\underline{P}$ onto $\underline{P}'$ which satisfies the following condition, for every $\alpha \in P$ and $\beta' \in P'$:*

- *if $g(\alpha) <' \beta'$, there is $\beta \in P$ s.t. β has finite V-grade in $\underline{K}$, $\alpha < \beta$ and $g(\beta) = \beta'$.*

□

5.1.12 Theorem (Strong ω-completeness Criterion)
Let L be a strongly ω-complete logic, let V be a finite set of propositional variables, let $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ be a V-separable and V-full model of L^V and let $\underline{P}'$ be a V-stable reduction of $\underline{P}$. Then $\underline{P}'$ is a frame for L.

Proof: Suppose that $\underline{P}'$ is not a frame for L. To prove the proposition, it suffices to prove that:

(1) For every $\underline{K}'' = \langle P'', \leq'', \rho'', \Vdash'' \rangle$ such that $\Gamma^V_{\underline{K}''}(\rho'') = \Gamma^V_{\underline{K}}(\rho)$, the frame $\underline{P}'' = \langle P'', \leq'', \rho'' \rangle$ is not a frame for L.

Indeed (1) implies that the L, V-saturated set $\Gamma^V_{\underline{K}}(\rho)$ cannot be realized in any Kripke model based on a frame for L, that is L is not strongly ω-complete (see Proposition 1.5.3 (iv)).
Let $\underline{K}''$ be as in (1), let $h : \underline{P}'' \to \underline{P}$ be as in Proposition 5.1.6 and let g be the p-morphism from $\underline{P}$ onto $\underline{P}'$ as in the definition of V-stable reduction. We know that h is "less than" a p-morphism, while g is "more than" a p-morphism; composing these two maps, we get:

(2) $f = g \circ h$ is a p-morphism from $\underline{P}''$ onto $\underline{P}'$

from which (1) follows.
Let $\alpha'', \beta'' \in P''$; $\alpha'' \leq'' \beta''$ implies $h(\alpha'') \leq h(\beta'')$, which implies $g(h(\alpha'')) \leq' g(h(\beta''))$, that is $f(\alpha'') \leq' f(\beta'')$.
Let $\alpha'' \in P''$ and suppose $f(\alpha'') <' \beta'$, that is $g(h(\alpha'')) <' \beta'$. By definition of g, there is $\beta \in P$ such that:
- β has finite V-grade in $\underline{K}$,
- $h(\alpha'') < \beta$ and $g(\beta) = \beta'$.
By definition of h, there is $\beta'' \in P''$ such that:
- $\alpha'' \leq'' \beta''$ and $h(\beta'') = \beta$.
Hence $f(\beta'') = g(h(\beta'')) = g(\beta) = \beta'$.
On the other hand, $f(\rho'') = g(h(\rho'')) = g(\rho) = \rho'$, hence f is also surjective and (2) is proved. □

5.2 Non strong ω-completeness of the logics $\mathbf{T}_n$

First of all, we prove that the logics of finite branching $\mathbf{T}_n$ are not ω-canonical (this result is proved also in [15] in a different way, even if the countermodel $\underline{K}$ is the same). Let us consider the frame $\underline{P} = \langle P, \leq, e_3 \rangle$ defined as in Figure 5.1. We point out that:

- $P = \{a_n, b_n, c_n, d_l, e_m, \alpha : n \geq 0,\ l \geq 2,\ m \geq 3\}$.

Moreover, for every $\delta \in P$ it holds that:

- $\alpha < \delta$ iff $\delta \in \{a_n, b_n, c_n : n \geq 0\}$
- $d_k < \delta$ iff $\delta \in \{a_n, b_j, c_j : 0 \leq n \leq k-1,\ 0 \leq j \leq k-2\}$
- $e_k < \delta$ iff $\delta \in \{a_n, b_n, c_n, d_l, e_m, \alpha : n \geq 0,\ l \geq k-1,\ m \geq k+1\}$.

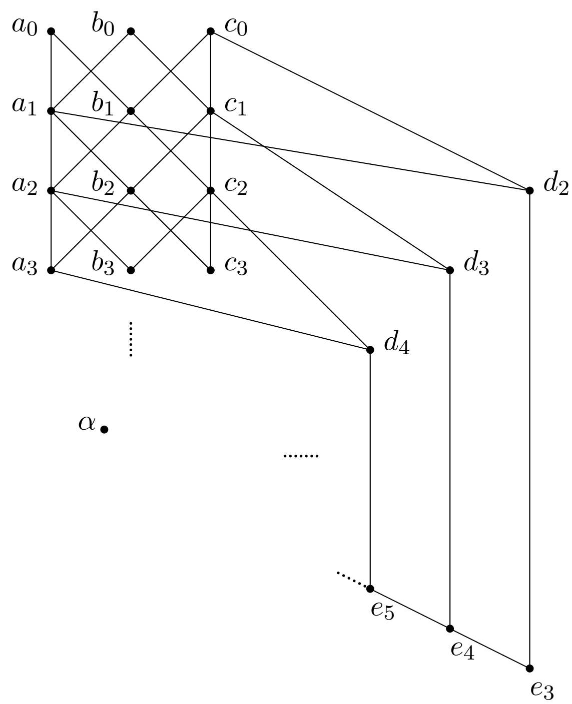

Figure 5.1: The model $\underline{K}$ of $\mathbf{T}_n$

The forcing relation is defined so that the following properties hold with respect to some finite set V of propositional variables.

- a_0, b_0, c_0 have V-grade 0.
- The points in the set $\{a_n, b_n, c_n, d_l, e_m, \alpha : n \geq 1,\ l \geq 2,\ m \geq 3\}$ are $\sim_0^V$ equivalent (clearly, they are not $\sim_0^V$ equivalent to any of the points a_0, b_0, c_0, otherwise these ones would not have V-grade 0).

We stress that in this example and in the further ones we are not interested in specifying in all details the forcing relation; on the other hand, it is not difficult to see that a forcing relation matching the above conditions with respect to some finite set V can be actually worked out.

5.2.1 Lemma *For every $n \geq 0$ the following holds.*

(i) a_n, b_n, c_n, d_n, e_n *(when defined) have V-grade n.*

(ii) The points $\{a_k, b_k, c_k, d_k, e_k : k \geq n+1\} \cup \{\alpha\}$ *are* $\sim_n^V$ *equivalent.*

(iii) α *has infinite V-grade.*

Proof: (i) and (ii) are proved, by induction on $n \geq 0$, by directly checking the definition of the relation $\sim_n^V$; (iii) is an immediate consequence of (ii). □

5.2.2 Proposition *$\underline{K}$ is V-separable and V-full.*

Proof: The V-separability is immediate, since the points of finite V-grade are V-separated from all the other points and we have only one point of infinite V-grade (see Proposition 5.1.8). In order to prove the V-fullness, we apply Proposition 5.1.10. Let $\{\beta_k\}_{k\geq 0}^V$ be a V-sequence of $\underline{K}$ not definitively constant; taking into account how the points of $\underline{K}$ are partitioned in $\sim_n^V$-classes, it is easy to check that the only cone of $\underline{K}$ containing $\{\beta_k\}_{k\geq 0}^V$ is $\underline{K}$ itself and α is the limit of such a V-sequence; thus $\underline{K}$ is V-full. □

5.2.3 Proposition *$\underline{K}$ is a model of $\mathbf{T}_n^V$, for every $n \geq 2$.*

Proof: Let $n \geq 2$ and suppose that $\underline{K}$ is not a model of $\mathbf{T}_n^V$. Then there is a V-formula H such that $H \in \mathbf{T}_n^V$ and H is not valid in $\underline{K}$; without loss of generality, we can assume that H is an instance of the axiom scheme of $\mathbf{T}_n$, that is:

$$H = \bigwedge_{i=0}^{n}((A_i \to \bigvee_{j\neq i} A_j) \to \bigvee_{j\neq i} A_j) \to \bigvee_{i=0}^{n} A_i$$

where all the A_i are V-formulas. Let $\delta \in P$ be such that $\delta \nVdash H$; we show that δ necessarily is one of the points e_k. If δ is one of the points a_n, b_n, c_n, d_m, with $n \geq 0$ and $m \geq 2$, then $\delta \Vdash H$, since the cone $\underline{P}_\delta$ generated by δ is a finite frame for the logic $\mathbf{T}_n$. Moreover, let r be the implicational complexity of H; since $\alpha \sim_r^V a_{r+1}$ and $a_{r+1} \Vdash H$, it follows that $\alpha \Vdash H$ as well. On the other hand, it is not the case that all the points e_k do not force H (for instance, since $e_{r+1} \sim_r^V a_{r+1}$, then $e_{r+1} \Vdash H$). Therefore we can assume that there is $k \geq 3$ such that:

- $e_k \nVdash H$;
- for every $\delta > e_k$, $\delta \Vdash H$.

This implies that:

- $e_k \Vdash \bigwedge_{i=0}^{n}(A_i \to \bigvee_{j\neq i} A_j) \to \bigvee_{j\neq i} A_j$;
- $e_k \nVdash \bigvee_{i=0}^{n} A_i$.

It follows that, for every $0 \leq i \leq n$, there is δ_i such that $e_k \leq \delta_i$, $\delta_i \Vdash A_i$ (hence $\delta_i \neq e_k$) and $\delta_i \nVdash \bigvee_{j\neq i} A_j$. Since $n \geq 2$ and e_k has only two immediate successors, one of them, let us call it β^*, sees at least two distinct such points δ_i; thus $\beta^* \nVdash \bigvee_{i=0}^{n} A_i$. On the other hand, by the above assumptions, $\beta^* \Vdash H$ and β^* forces the antecedent of H, hence $\beta^* \Vdash \bigvee_{i=0}^{n} A_i$ must hold, a contradiction. We can conclude that the initial assumption is false and thus the proposition is proved. □

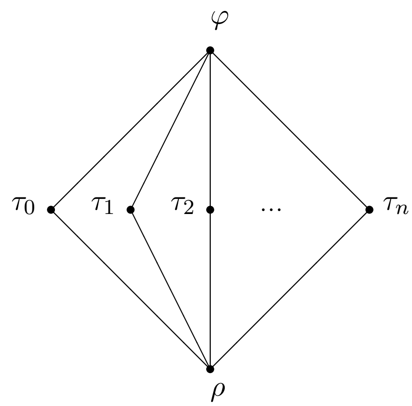

Figure 5.2: The frame $\underline{\tilde{P}}_n$

5.2.4 Theorem $\mathbf{T}_n$ *is not strongly* ω*-complete, for every* $n \geq 2$.

Proof: Let $n \geq 2$ and let us consider the finite frame $\underline{\tilde{P}}_n$ in Figure 5.2. We prove that $\underline{\tilde{P}}_n$ is a V-stable reduction of $\underline{P}$. At this aim, we can take the p-morphism g from $\underline{P}$ onto $\underline{\tilde{P}}_n$ defined as follows:

- $g(e_k) = \rho$ for $k \geq 3$.

- $g(d_{(n+1)k+j}) = \tau_j$ for $0 \leq j \leq n$ and $k \geq 0$.

- $g(a_k) = g(b_k) = g(c_k) = g(\alpha) = \varphi$ for $k \geq 0$.

Since $\underline{K}$ is a V-separable and V-full model of $\mathbf{T}_n^V$ and $\underline{\tilde{P}}_n$ is not a frame for $\mathbf{T}_n$ ($\underline{\tilde{P}}_n$ is finite and ρ has $n+1$ immediate successors), by the Strong ω-completeness Criterion we can conclude that $\mathbf{T}_n$ is not strongly ω-complete. □

In the previous proof we have implicitly showed that the V-separable and V-full model $\underline{K}$ of $\mathbf{T}_n^V$ is based on a frame $\underline{P}$ which is not a frame for $\mathbf{T}_n$. We remark that, in order to obtain countermodels of this kind, we are obliged to insert points of infinite depth. Indeed, if $\underline{K}$ is a V-separable model of L^V, any cone $\underline{K}_\delta$ of $\underline{K}$ generated by some point δ of finite depth is finite, hence it is based on a frame for L. Besides this, if $\underline{K}$ is also V-full, then there are points β, we call them *infinite maximal* points, such that β has infinite depth and any point $\delta > \beta$ has finite depth (see the Appendix). We will prove (Corollary A.0.5) that, for such β, the cone $\underline{K}_\beta$ of $\underline{K}$ is based on a frame for L as well (for instance, in the frame $\underline{P}$ of the previous proof, α is an infinite maximal point and one can directly check that $\underline{P}_\alpha$ is a frame for $\mathbf{T}_n$). This is the reason why, in this kind of proofs, we have to build very deep models.

5.3 Non extensive ω-canonicity of the logic KP

We now show an application of these techniques to disprove the extensive ω-canonicity of the logic **KP** (this improves [15], where it is stated that **KP** is not extensively canonical). The idea is to define an (infinite) well V-separable model of $\mathbf{KP}^V$, for some finite V, whose frame is not a frame for **KP**. At this aim, let us consider the frame $\underline{P} = \langle P, \leq, r_0 \rangle$ of the Figure 5.3. More precisely, $\underline{P}$ satisfies the following

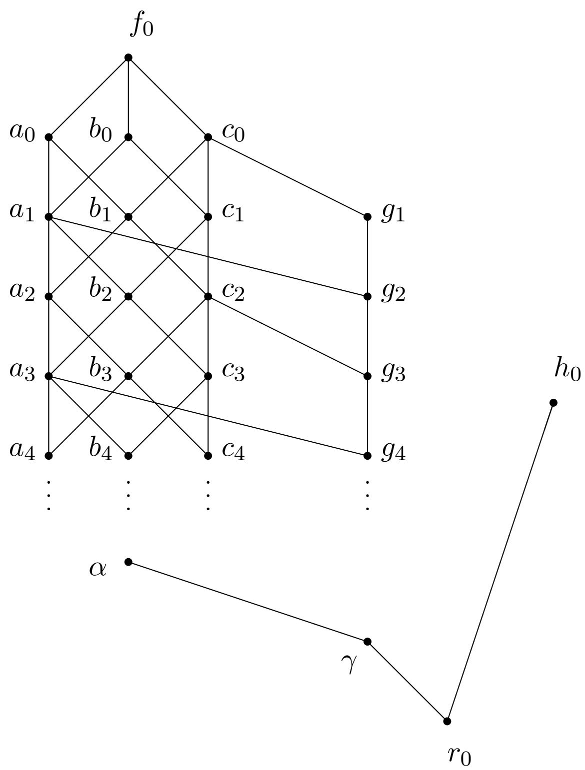

Figure 5.3: The model $\underline{K}$ of **KP**

conditions:

- $P = \{a_n, b_n, c_n, g_m, f_0, h_0, r_0, \alpha, \gamma : n \geq 0, m \geq 1\}$.
- $\alpha < \delta$ iff $\delta \in \{f_0, a_n, b_n, c_n : n \geq 0\}$.
- $\gamma < \delta$ iff either $\alpha \leq \delta$ or $\delta = g_k$ for some $k \geq 1$.
- $\delta < h_0$ iff $\delta = r_0$.

We define a model $\underline{K} = \langle P, \leq, r_0, \Vdash \rangle$ based on $\underline{P}$, where the forcing relation satisfies the following conditions with respect to some finite set V of propositional variables.

- $f_0, a_0, b_0, c_0, h_0, r_0$ have V-grade 0.
- The other distinct equivalence classes, with respect to the relation $\sim_0^V$, having more than one element are:
  - $\{a_n, b_n, c_n : n \geq 1\} \cup \{\alpha\}$;
  - $\{g_n : n \geq 1\} \cup \{\gamma\}$.

It is not difficult to prove that:

5.3.1 Lemma *For every $n \geq 1$ the following holds.*

(i) a_n, b_n, c_n, g_n *have V-grade n.*

(ii) The distinct equivalence classes with respect to the relation $\sim_n^V$ having more than one element are:

- $\{a_k, b_k, c_k : k \geq n+1\} \cup \{\alpha\}$;
- $\{g_k : k \geq n+1\} \cup \{\gamma\}$.

(iii) α *and* γ *have infinite V-grade.*

□

5.3.2 Proposition $\underline{K}$ *is a V-separable and V-full model of* $\mathbf{KP}^V$*.*

Proof: To prove the V-separability and the V-fullness of $\underline{K}$, we can proceed as in Proposition 5.2.2 (we point out that, in this case, we have two points of infinite V-grade which are surely V-separated, in fact they are not even $\sim_0^V$ equivalent). Moreover, it is immediate to see that $\underline{P}$ is a frame for the logic **KP** (see Proposition 2.4.2), hence $\underline{K}$ is a model of $\mathbf{KP}^V$. □

Now, let us consider the model $\underline{K}' = \langle P', \leq', r_0' \Vdash' \rangle$ obtained from $\underline{K}$ by deleting the point γ; more formally, $\underline{K}'$ is defined as follows (see Figure 5.4).

- $P' = \{\delta' : \delta \in P \text{ and } \delta \neq \gamma\}$.
- $\delta_1' <' \delta_2'$ iff $\delta_1 < \delta_2$.
- $\delta' \Vdash' p$ iff $\delta \Vdash p$.

It is immediate to prove that, for every $\delta' \in P'$ and every $n \geq 0$, $\delta' \sim_n^V \delta$. This allows us to state that:

5.3.3 Proposition $\underline{K}'$ *is a well V-separable model of* $\mathbf{KP}^V$*.* □

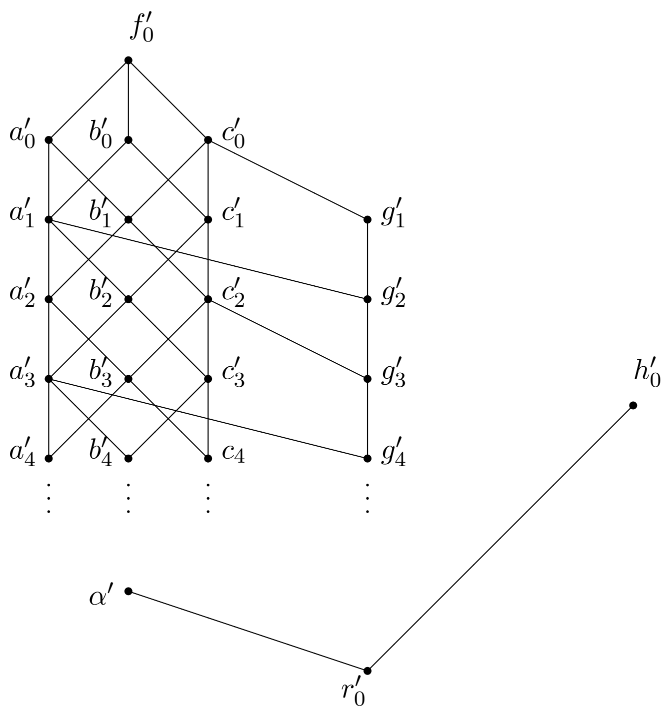

Figure 5.4: Model $\underline{K}'$ of **KP**

It follows that:

5.3.4 Theorem *The logic* **KP** *is not extensively ω-canonical.*

Proof: We prove that the frame $\underline{P}'$ of the well V-separable model $\underline{K}'$ of $\mathbf{KP}^V$ does not satisfy the necessary condition relative to the frames for **KP** with enough final points (see Proposition 2.4.2). As a matter of fact, let us take r'_0, α' and g'_1; then, there is not any $\delta' \in P'$ such that $r'_0 \leq' \delta'$, $\delta' \leq' \alpha'$, $\delta' \leq' g'_1$ and $\mathrm{Fin}(\delta') = \{f'_0\}$, thus $\underline{P}'$ is not a frame for **KP** and **KP** is not extensively ω-canonical. □

We point out that, according to the fact that **KP** is ω-canonical, $\underline{K}'$ is not V-full, since the V-sequence $\{g'_n\}_{n\geq 1}$ has not V-limit in $\underline{K}'$ (indeed, it corresponds to the point γ of $\underline{K}$).

5.4 Strong ω-completeness of the logics in one variable

We continue the analysis of the logics in one variable. First of all, we give a condition about the infinite models of such logics. Even if any frame $\underline{P}$ not belonging to $\mathrm{Spl}(\underline{P}_{\sigma_m})$ is not a frame for $\mathbf{NL}_{m+1}$, this fact does not prevent us from defining V-separable models $\underline{K}$ of $\mathbf{NL}^V_{m+1}$, with V finite, based on such a frame $\underline{P}$. The idea beyond this is that the possible p-morphisms from the generated subframes of

$\underline{P}$ onto $\underline{P}_{\sigma_m}$ do not allow to separate, by means of V-formulas, the preimages of σ_1 by the preimages of σ_3 (clearly, $\underline{P}$ must be infinite). The following proposition is in order.

5.4.1 Proposition *Let $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ be a Kripke model and let V be a finite set of propositional variables.*

(i) $\underline{K}$ is a model of $\mathbf{NL}^V_{m+1}$, for $m \geq 5$, if and only if the following condition holds:

(†) for every generated subframe $\underline{P}'$ of $\underline{P}$, for every p-morphism f from $\underline{P}'$ onto $\underline{P}_{\sigma_m}$, for every $k \geq 0$, there are δ_1 and δ_2 in $\underline{P}'$ such that $\delta_1 \sim^V_k \delta_2$, $f(\delta_1) = \sigma_1$ and $f(\delta_2) \neq \sigma_1$.

(ii) $\underline{K}$ is a model of $\mathbf{NL}^V_{n+1,n+2}$, for $n \geq 2$, if and only if the following condition holds:

(††) for every generated subframe $\underline{P}'$ of $\underline{P}$, for every p-morphism f from $\underline{P}'$ onto $\underline{P}_{\sigma_{n,n+1}}$, for every $k \geq 0$, there are δ_1 and δ_2 in $\underline{P}'$ such that $\delta_1 \sim^V_k \delta_2$, $f(\delta_1) = \sigma_1$ and $f(\delta_2) \neq \sigma_1$.

Proof:
(i) Suppose that, for some $m \geq 5$, $\underline{K}$ does not satisfy (†); then there is a generated subframe $\underline{P}'$ of $\underline{P}$ and a p-morphism f from $\underline{P}'$ onto $\underline{P}_{\sigma_m}$ such that, for some $n \geq 0$, it holds that:

(*) for every δ_1, δ_2 in $\underline{P}'$, $\delta_1 \sim^V_n \delta_2$ and $f(\delta_1) = \sigma_1$ implies $f(\delta_2) = \sigma_1$.

Let $\alpha_1, \ldots, \alpha_j$ be a finite list of points of $\underline{P}'$ such that:

- $f(\alpha_k) = \sigma_1$ for every $1 \leq k \leq j$;
- for every β in $\underline{P}'$ such that $f(\beta) = \sigma_1$, $\beta \sim^V_{n+1} \alpha_k$ for some $1 \leq k \leq j$.

By Lemma 5.1.3, there are some V-formulas $H_1, \ldots, H_j$ such that, for $1 \leq k \leq j$, the following holds:

- for every β in $\underline{P}'$, $\beta \Vdash H_k$ iff $\alpha_k \preccurlyeq^V_{n+1} \beta$.

Let H be the V-formula $H_1 \vee \cdots \vee H_j$. We prove that:

(**) for every β in $\underline{P}'$, $\beta \Vdash H$ iff $f(\beta) = \sigma_1$.

If $\beta \Vdash H$, then $\beta \Vdash H_k$ for some $1 \leq k \leq j$, hence $\alpha_k \preccurlyeq^V_{n+1} \beta$. By definition of $\preccurlyeq^V_{n+1}$, there is α in $\underline{P}'$ such that $\alpha_k \leq \alpha$ and $\alpha \sim^V_n \beta$; since $f(\alpha_k) = \sigma_1$, it follows that $f(\alpha) = \sigma_1$ hence, by (*), $f(\beta) = \sigma_1$. Suppose now that $f(\beta) = \sigma_1$ and let k be such that $\beta \sim^V_{n+1} \alpha_k$; then $\beta \Vdash H_k$, that is $\beta \Vdash H$; thus (**) is proved.
Let δ be any point of $\underline{P}'$ such that $f(\delta) = \sigma_m$. Since in the model $\underline{K}_\omega$ (defined in Section 2.5) σ_1 forces p, σ_m does not force the $\{p\}$-formula $\boldsymbol{nf}_{m+1}$ and f is a p-morphism from $\underline{P}'$ onto $\underline{P}_{\sigma_m}$, by (**) δ does not force the instance $\boldsymbol{nf}_{m+1}(H)$ of

$\boldsymbol{nf}_{m+1}$ obtained by replacing p with the V-formula H; we can conclude that $\underline{K}$ is not a model of $\mathbf{NL}^V_{m+1}$.
Conversely, suppose that $\underline{K}$ is not a model for $\mathbf{NL}^V_{m+1}$. We can assume that, for some V-formula H, the instance $\boldsymbol{nf}_{m+1}(H)$ of the axiom schema of $\mathbf{NL}_{m+1}$ is not valid in $\underline{K}$. Let $\alpha \in P$ be such that $\alpha \not\Vdash \boldsymbol{nf}_{m+1}(H)$, that is:

$$\alpha \not\Vdash \boldsymbol{nf}_m(H) \to \boldsymbol{nf}_{m-2}(H) \vee \boldsymbol{nf}_{m-3}(H).$$

Then there is $\beta \geq \alpha$ such that $\beta \Vdash \boldsymbol{nf}_m(H)$, $\beta \not\Vdash \boldsymbol{nf}_{m-2}(H)$ and $\beta \not\Vdash \boldsymbol{nf}_{m-3}(H)$. Let us define a map on the points $\delta \geq \beta$ as follows:

- $f(\delta) = \sigma_k$ iff $\delta \Vdash \boldsymbol{nf}_k(H)$, $\delta \not\Vdash \boldsymbol{nf}_{k-2}(H)$, $\delta \not\Vdash \boldsymbol{nf}_{k-3}(H)$, for every $4 \leq k \leq m$;
- $f(\delta) = \sigma_3$ iff $\delta \Vdash \boldsymbol{nf}_3(H)$ and $\delta \not\Vdash \boldsymbol{nf}_1(H)$;
- $f(\delta) = \sigma_2$ iff $\delta \Vdash \boldsymbol{nf}_2(H)$;
- $f(\delta) = \sigma_1$ iff $\delta \Vdash \boldsymbol{nf}_1(H)$;

where $\boldsymbol{nf}_1(H)$ coincides with H. It is easy to check that f is a p-morphism from $\underline{P}_\beta$ onto $\underline{P}_{\sigma_m}$. Let $r \geq 0$ be the implicational complexity of H. By definition of f, for every δ_1, δ_2 in $\underline{P}_\beta$ it holds that:

$$\delta_1 \sim^V_r \delta_2 \implies f(\delta_1) = \sigma_1 \text{ iff } f(\delta_2) = \sigma_1.$$

This means that (†) does not hold.
(ii) Is proved as (i). □

Note that, if $\underline{K}$ is V-separable and $k \geq 2$, the hypothesis $\delta_1 \sim^V_k \delta_2$ and $f(\delta_1) = \sigma_1$ implies that $f(\delta_2) \in \{\sigma_1, \sigma_3\}$ (indeed, $\mathrm{Fin}(\delta_1) = \mathrm{Fin}(\delta_2)$).

5.4.1 The ω-canonical logics in one variable

So far we have proved that all the logics in one variable, except few of them (namely **Cl**, **Jn**, $\mathbf{NL}_{3,4}$ and $\mathbf{NL}_{4,5}$, which are canonical), are not strongly complete. Now we improve the result by showing that all the logics in one variable, except few of them (namely, the four canonical ones plus the logics **St**, **Ast**, $\mathbf{NL}_{5,6}$ and $\mathbf{NL}_{6,7}$), are not strongly ω-complete. Let us start with the trivial case of the logic $\mathbf{NL}_6 = \mathbf{St}$.

5.4.2 Theorem *The logic* **St** *is extensively ω-canonical.*

Proof: Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a well V-separable model of $\mathbf{St}^V$, for some finite V, and suppose that $\underline{P}$ is not a frame for **St**. Then there is a p-morphism f from some generated subframe $\underline{P}'$ of $\underline{P}$ onto $\underline{P}_{\sigma_5}$. Since $\underline{K}$ has finitely many final points and $\underline{K}$ is V-separable, there is a V-formula H of implicational complexity 1 such that:

- $\delta \Vdash H$ if δ is final and $f(\delta) = \sigma_1$;
- $\delta \Vdash \neg H$ if δ is final and $f(\delta) = \sigma_2$.

Let $\alpha \in P$ be such that $f(\alpha) = \sigma_5$; it is easy to see that $\alpha \nVdash \boldsymbol{nf}_6(H)$, which is an instance of the axiom schema of **St**. Since such a formula belongs to $\mathbf{St}^V$, we get a contradiction. Thus f does not exist and **St** is extensively ω-canonical. $\square$

The proof of ω-canonicity of the remaining logics is not trivial, and an essential use of the hypothesis of V-fullness is needed.

5.4.3 Theorem *The logics* **Ast**, $\mathbf{NL}_{5,6}$ *and* $\mathbf{NL}_{6,7}$ *are ω-canonical.*

Proof: We prove the theorem only for the logic $\mathbf{NL}_{6,7}$ (the other cases are similar[1]). Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a V-separable and V-full model of $\mathbf{NL}^V_{6,7}$, for some finite V, and suppose that $\underline{P}$ is not a frame for $\mathbf{NL}_{6,7}$. Then there is a p-morphism f from some generated subframe $\underline{P}'$, contained in some cone $\underline{P}_{\tilde{\alpha}}$ of $\underline{P}$, onto $\underline{P}_{\sigma_{5,6}}$. Let H be the V-formula defined in the proof of Theorem 5.4.2; then, for every δ in $\underline{P}'$, it holds that:

- $\delta \Vdash \neg H \vee \neg\neg H$ iff $f(\delta) \in \{\sigma_1, \sigma_2, \sigma_3\}$

(note that $\neg H \vee \neg\neg H$ has implicational complexity 3). We can apply Proposition 1.6.3 and we can state that:

(a) Let $\alpha \in P$ be such that $f(\alpha) = \tilde{\sigma}$, with $\tilde{\sigma} \in \{\sigma_4, \sigma_5\}$. Then there is β such that $\alpha \leq \beta$, $f(\beta) = \tilde{\sigma}$ and, for every $\delta > \beta$, $f(\delta) \neq \tilde{\sigma}$.

Thus we can take $\alpha, \beta, \alpha', \beta' \geq \tilde{\alpha}$, P° and $\mathscr{G}_{MAX}$ as follows.

- $f(\alpha) = \sigma_5$ and, for all $\delta > \alpha$, $f(\delta) \neq \sigma_5$;
- $f(\alpha') = \sigma_6$;
- $\alpha < \beta, \alpha' < \beta'$ and $f(\beta) = f(\beta') = \sigma_3$;
- $P^\circ = \{\delta \in P : \alpha \leq \delta \text{ or } \alpha' \leq \delta\}$;
- $\mathscr{G}_{MAX} = \{\gamma \in P^\circ : f(\gamma) = \sigma_4 \quad \text{and, for all } \delta > \gamma,\ f(\delta) \neq \sigma_4\}$.

Clearly the generated submodel $\underline{K}^\circ = \langle P^\circ, \leq, \Vdash \rangle$ of $\underline{K}$ is a V-separable and V-full model of $\mathbf{NL}^V_{6,7}$. We prove the following facts.

(b) For every V-sequence $\{\gamma_k\}^V_{k\geq 0} \subseteq \mathscr{G}_{MAX}$, the limit of $\{\gamma_k\}^V_{k\geq 0}$ belongs to $\mathscr{G}_{MAX}$.

In fact, by the V-fullness of $\underline{K}^\circ$, $\{\gamma_k\}^V_{k\geq 0}$ has a limit $\gamma^* \in P^\circ$; since $\alpha' \leq \gamma^*$ and $\gamma^* \nVdash \neg H \vee \neg\neg H$, it follows that $f(\gamma^*) \in \{\sigma_4, \sigma_6\}$. Therefore there is γ' such that $\gamma^* \leq \gamma'$ and $\gamma' \in \mathscr{G}_{MAX}$. Suppose that $\gamma^* \neq \gamma'$; by the V-separability of $\underline{K}^\circ$, there is $k \geq 3$ such that $\gamma^* \not\sim^V_k \gamma'$. Let $\delta \geq \gamma_{k+1}$ be such that $\delta \sim^V_k \gamma'$; since $\delta \nVdash \neg H \vee \neg\neg H$ (indeed $\gamma' \nVdash \neg H \vee \neg\neg H$ and $k \geq 3$), by the maximality of γ_{k+1} it holds that $\delta = \gamma_{k+1}$, that is $\gamma' \sim^V_k \gamma^*$, a contradiction; hence $\gamma^* = \gamma'$ and $\gamma^* \in \mathscr{G}_{MAX}$.

[1] The case of **Ast** is also treated in [15] in a different way.

(c) There is $n \geq 0$ such that, for every $\gamma \in \mathscr{G}_{MAX}$, for every $\delta \in P^\circ$ s.t. $\delta \sim_n^V \beta$, it holds that $\gamma \not< \delta$.

Suppose that (c) does not hold; then, for every $n \geq 0$, there are $\gamma_n \in \mathscr{G}_{MAX}$ and $\beta_n \sim_n^V \beta$ such that $\gamma_n < \beta_n$. Starting from the points γ_n, we can extract a V-sequence $\{\gamma'_k\}_{k\geq 0}^V$ contained in $\mathscr{G}_{MAX}$ in the following way: γ'_0 is chosen in a $\sim_0^V$ class containing infinitely many γ_n (that is, in such a class there occur infinitely many indexes n of non necessarily distinct points γ_n); γ'_{k+1} is chosen in the $\sim_k^V$ class $[\gamma'_k]_{\sim_k^V}$ (which, inductively, contains infinitely many γ_n) in such a way that $[\gamma'_{k+1}]_{\sim_{k+1}^V}$ contains infinitely many points γ_n as well (this is a kind of Bolzano-Weierstrass construction which can be carried out by the fact that there are only finitely many $\sim_k^V$ equivalence classes). Let γ^* be the limit of such a V-sequence; then $\Gamma^{V,k}(\gamma^*) \subseteq \Gamma^{V,k}(\beta)$ for every $k \geq 0$, hence $\Gamma^V(\gamma^*) \subseteq \Gamma^V(\beta)$ and, by the well V-separability of $\underline{K}^\circ$, $\gamma^* \leq \beta$. This yields a contradiction, since, by (b), $\gamma^* \in \mathscr{G}_{MAX}$ and $f(\beta) = \sigma_3$; therefore (c) holds. Similarly, we can prove:

(d) There is $m \geq 0$ such that, for every $\gamma \in \mathscr{G}_{MAX}$, for every $\delta \in P^\circ$ s.t. $\delta \sim_m^V \beta'$, it holds that $\gamma \not< \delta$.

Let n and m be as in (c) and in (d) respectively, let $r = \max(n, m, 2)$ and consider the set

$$\mathscr{B} = \{\delta \in P^\circ : \delta \sim_r^V \beta \text{ or } \delta \sim_r^V \beta'\}.$$

Since $r \geq 2$, every point $\delta \in \mathscr{B}$ is not final and $\mathrm{Fin}(\delta) \subseteq \mathrm{Fin}(\beta) \cup \mathrm{Fin}(\beta')$. By (c) and (d), it holds that:

(e) For every $\gamma \in \mathscr{G}_{MAX}$ and every $\delta \in \mathscr{B}$, $\gamma \not< \delta$.

We define a map $g : \underline{P}^\circ \to \underline{P}_{\sigma_{5,6}}$ as follows.

- $g(\delta) = \sigma_1$ iff $f(\delta) \in \{\sigma_1, \sigma_3\}$ and, for all $\delta' \in \mathscr{B}$, $\delta \not\leq \delta'$.
- $g(\delta) = \sigma_2$ iff $f(\delta) = \sigma_2$.
- $g(\delta) = \sigma_3$ iff $f(\delta) \in \{\sigma_1, \sigma_3\}$ and there is $\delta' \in \mathscr{B}$ such that $\delta \leq \delta'$.
- $g(\delta) = \sigma_4$ iff there is $\gamma \in \mathscr{G}_{MAX}$ such that $\delta \leq \gamma$ and, for all $\delta' \in \mathscr{B}$, $\delta \not\leq \delta'$.
- $g(\delta) = \sigma_5$ iff $\delta = \alpha$.
- $g(\delta) = \sigma_6$ iff there is $\delta' \in \mathscr{B}$ s.t. $\delta \leq \delta'$ and there is $\gamma \in \mathscr{G}_{MAX}$ s.t. $\delta \leq \gamma$.

Note in particular that $g(\delta) = \sigma_3$ for every $\delta \in \mathscr{B}$; by (e), $g(\gamma) = \sigma_4$ for every $\gamma \in \mathscr{G}_{MAX}$, $g(\alpha') = \sigma_6$. Taking into account the previous statements, it is not difficult to prove that g is a p-morphism from $\underline{P}^\circ$ onto $\underline{P}_{\sigma_{5,6}}$. Since $\underline{K}_{\tilde{\alpha}}$ is a model of $\mathbf{NL}_{6,7}^V$ with root $\tilde{\alpha}$, by Condition (††) of Proposition 5.4.1 there are δ and δ' in $\underline{P}^\circ$ such that $\delta \sim_{r+1}^V \delta'$, $g(\delta) = \sigma_1$ and $g(\delta') \neq \sigma_1$, that is $g(\delta') = \sigma_3$. By definition of g, there is $\beta^* \in \mathscr{B}$ such that $\delta' \leq \beta^*$; being $\delta \sim^V_{r+1} \delta'$, there is $\delta^* \geq \delta$ such that $\delta^* \sim^V_r \beta^*$. This implies that $\delta^* \in \mathscr{B}$, hence $g(\delta^*) = \sigma_3$, which is absurd. We can conclude that the p-morphism f cannot exist, therefore $\underline{P}$ is a frame for $\mathbf{NL}_{6,7}$ and $\mathbf{NL}_{6,7}$ is ω-canonical. □

We stress that in the previous theorem we have heavily used the V-fullness hypothesis and such an hypothesis cannot be avoided, due to the fact that such logics are not extensively ω-canonical (we will only treat the nontrivial case of the logic $\mathbf{NL}_{5,6}$ at the end of this chapter; for **Ast** see [15], and the case of $\mathbf{NL}_{6,7}$ is similar). Thus the analysis of the strongly ω-complete logics is completed.

The next step is to prove that all the other logics are not strongly ω-complete. We can proceed as in the case of strong completeness: firstly we disprove the strong ω-completeness of $\mathbf{NL}_8$ by exhibiting a model $\underline{K} = \langle P, \leq, r_0, \Vdash \rangle$ such that, for some finite V, $\underline{K}$ is a model of $\mathbf{NL}_8^V$ and $\underline{P}_{\sigma_7}$ is a V-stable reduction of $\underline{P} = \langle P, \leq, r_0 \rangle$. This proof (just in the case of **St**) can be easily extended to the infinitely many logics $\mathbf{NL}_{m+1}$ and $\mathbf{NL}_{n+1,n+2}$ for $m \geq 9$ and $n \geq 6$ (namely, the logics in one variable strictly contained in $\mathbf{NL}_8$), in virtue of the fact that the frame $\underline{P}_{\sigma_7}$ is a cone of the frames $\underline{P}_{\sigma_m}$ and $\underline{P}_{\sigma_{n,n+1}}$. It is left out the logic $\mathbf{NL}_9$, which (as **Ast**) must be treated apart.

5.4.2 The logic $\mathbf{NL}_8$

The construction of the countermodel $\underline{K}$ for $\mathbf{NL}_8$ is rather complex, thus we proceed by degrees. First of all, we consider the "tower" of points a_k, b_k, c_k $(k \geq 0)$ of root α in Figure 5.5. Starting from this frame, we define the sequences of points

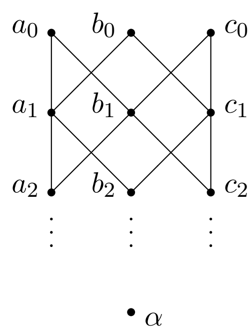

Figure 5.5: The points a_k, b_k, c_k, α

$d_3, d_4, d_5, \ldots$ and $g_1, g_2, \ldots, \gamma$ in the following way (see Figure 5.6).

- The immediate successors of d_k are a_{k-1} and c_{k-2} for every $k \geq 3$.
- The immediate successors of g_1 are a_0, b_0, c_0, f_0, where f_0 is a new final point.
- The immediate successors of g_{2k} are g_{2k-1} and a_{2k-1} for every $k \geq 1$.

- The immediate successors of g_{2k+1} are g_{2k} and c_{2k} for every $k \geq 1$.

- α is an immediate successor of γ; moreover, for every β, $\gamma < \beta$ iff $\alpha \leq \beta$ or $\beta = f_0$ or $\beta = g_n$ for some $n \geq 1$.

- $\alpha \not\leq d_k$, $\gamma \not\leq d_k$ and $\alpha \not\leq g_n$ for every $k \geq 3$ and $n \geq 1$.

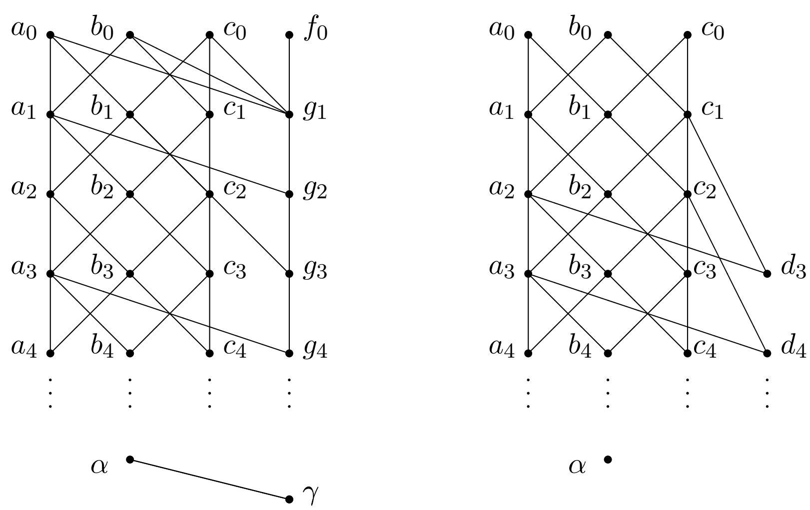

Figure 5.6: The sequences of points g_k and d_k

We now define the sequences $e_4, e_5, \ldots, \varepsilon$ and $h_4, h_5, \ldots, \eta$ (see Figure 5.7), which satisfy the following properties for every $k \geq 4$.

- The immediate successors of e_k are d_k and f_0.

- The immediate successors of h_k are d_{k-1} and e_k.

- The immediate successors of ε are α and f_0.

- The only immediate successor of η is ε (hence $\eta \not< d_j$ for every $j \geq 3$; a fortiori, $\eta \not< e_l$ and $\eta \not< h_l$ for every $l \geq 4$).

Finally, the root r_0 has, as immediate successors, all the points h_k ($k \geq 4$), γ and η (see Figure 5.8 for a global picture of $\underline{K}$).
The forcing relation is defined so that the following properties hold with respect to some finite set V of propositional variables.

- a_0, b_0, c_0, f_0, r_0 have V-grade 0.

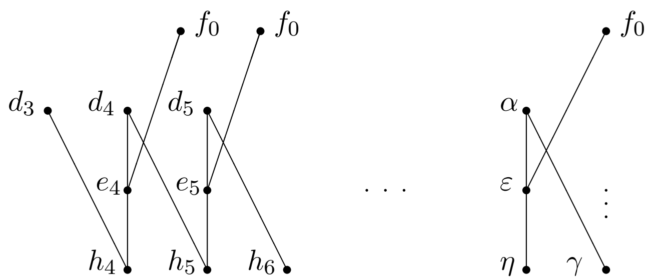

Figure 5.7: The sequences of points e_k and h_k

- The distinct equivalence classes with respect to the relation $\sim_0^V$ having more than one element are:
  - $\{a_k, b_k, c_k : k \geq 1\} \cup \{d_k : k \geq 3\} \cup \{\alpha\}$
  - $\{e_k : k \geq 4\} \cup \{\varepsilon\}$
  - $\{g_k : k \geq 1\} \cup \{\gamma\}$
  - $\{h_k : k \geq 4\} \cup \{\eta\}$.

It is indeed a routine task to find such a finite V and to define the forcing such that the above properties hold.

5.4.4 Lemma *For every $n \geq 0$ the following holds.*

(i) $a_n, b_n, c_n, d_n, g_n, e_n, h_n$ *(when defined) have V-grade n.*

(ii) The distinct equivalence classes with respect to the relation $\sim_n^V$ having more than one element are:
- $\{a_k, b_k, c_k, d_k : k \geq n+1\} \cup \{\alpha\}$
- $\{e_k : k \geq n+1\} \cup \{\varepsilon\}$
- $\{g_k : k \geq n+1\} \cup \{\gamma\}$
- $\{h_k : k \geq n+1\} \cup \{\eta\}$.

(iii) $\alpha, \gamma, \varepsilon, \eta$ *have infinite V-grade.*

Proof: (i) and (ii) are proved, by induction on $n \geq 0$, by directly checking the definitions; (iii) is an immediate consequence of (ii). □

5.4.5 Proposition *$\underline{K}$ is V-separable and V-full.*

Proof: To prove the V-separability, we can apply Proposition 5.1.8, observing that the points of infinite V-grade are pairwise V-separated since they belong to different $\sim_0^V$ classes. In order to prove the V-fullness, we apply Proposition 5.1.10. Let $\{\beta_k\}_{k\geq 0}^V$ be a non definitively constant V-sequence contained in some cone $\underline{K}_\delta$ of $\underline{K}$. Then one of the following facts holds:

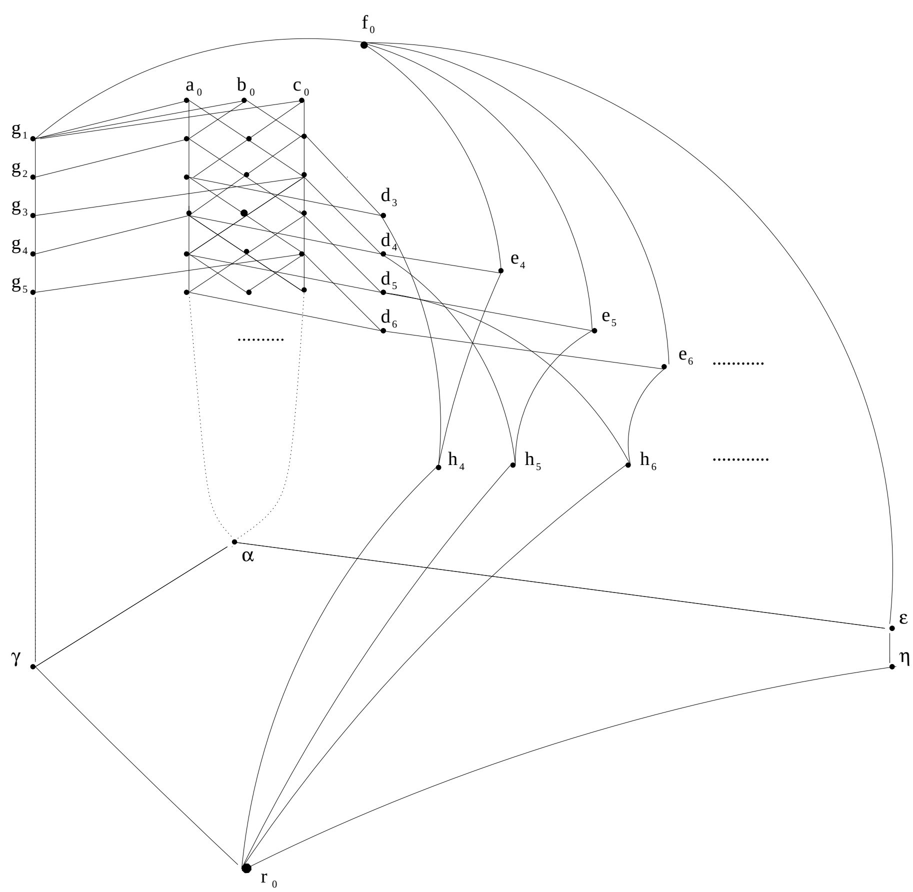

Figure 5.8: The model $\underline{K}$

(I) $\{\beta_k : k \geq 0\} \subseteq \{a_k, b_k, c_k : k \geq 0\} \cup \{d_k : k \geq 3\} \cup \{\alpha\}$

(II) $\{\beta_k : k \geq 0\} \subseteq \{e_k : k \geq 4\} \cup \{\varepsilon\}$

(III) $\{\beta_k : k \geq 0\} \subseteq \{g_k : k \geq 1\} \cup \{\gamma\}$

(IV) $\{\beta_k : k \geq 0\} \subseteq \{h_k : k \geq 4\} \cup \{\eta\}$.

We have to show that $\{\beta_k\}^V_{k\geq 0}$ has a limit β in $\underline{K}_\delta$. Suppose that (I) holds. Then α is the limit of $\{\beta_k\}^V_{k\geq 0}$; moreover, since $\{\beta_k\}^V_{k\geq 0}$ contains infinitely many distinct points, necessarily $\delta \leq \alpha$, and this concludes the proof. The other cases are similar and the limits are $\varepsilon, \gamma, \eta$ respectively. □

The most delicate question lies in proving that $\underline{K}$ is a model of $\mathbf{NL}^V_8$ since, in order to apply Proposition 5.4.1, we have to take into account *all* the possible p-morphisms from any generated subframe of $\underline{P}$ onto $\underline{P}_{\sigma_7}$. This is treated in next lemma.

5.4.6 Lemma *Let $\underline{P}'$ be a generated subframe of $\underline{P}$ and let f be a p-morphism from $\underline{P}'$ onto $\underline{P}_{\sigma_7}$. Then:*

(i) $f(\delta) = \sigma_7$ if and only if $\delta = r_0$ (hence $\underline{P}'$ coincides with $\underline{P}$).

(ii) $f(a_k) = \sigma_1$ for every $k \geq 0$.

(iii) There is $n \geq 3$ such that $f(d_k) = \sigma_3$ for every $k \geq n$.

Proof: Let $\underline{P}'$ be a generated subframe of $\underline{P}$ such that there is a p-morphism f from $\underline{P}'$ onto $\underline{P}_{\sigma_7}$. We firstly observe that:

($\star$) for every $\delta \in \{f_0, a_k, b_k, c_k, d_j, g_h : k \geq 0, j \geq 3, h \geq 1\}$, $f(\delta) \neq \sigma_5$.

This fact can be easily verified; as a matter of fact one can observe that the final points of the finite cone $\underline{P}_\delta$ of $\underline{P}$ are prefinally connected in $\underline{P}_\delta$ (in the sense of [8]), thus $\underline{P}_\delta$ is a frame for **St** and $f(\delta) \neq \sigma_5$. It follows that $\{a_0, b_0, c_0, f_0\} \subseteq \mathrm{Dom}(f)$, hence $\{f(a_0), f(b_0), f(c_0), f(f_0)\} \subseteq \{\sigma_1, \sigma_2\}$.

(1) $f(a_0) = f(b_0) = f(c_0)$.

If (1) does not hold, by the above remarks we get:

- $\{a_2, b_2, c_2\} \subseteq \mathrm{Dom}(f)$ and $f(a_2) = f(b_2) = f(c_2) = \sigma_4$.

This implies that σ_5 has not preimage in $\underline{P}'$, in fact:

- if $\delta \in \{a_0, b_0, c_0, a_1, b_1, c_1, g_1, g_2\}$ then, by ($\star$), $f(\delta) \neq \sigma_5$;
- in all the other cases, $\delta \leq a_2$ or $\delta \leq b_2$ or $\delta \leq c_2$, hence $f(\delta) \neq \sigma_5$.

From (1), it follows that:

(2) $f(a_0) = f(b_0) = f(c_0) = \sigma_1$ and $f(f_0) = \sigma_2$.

Otherwise, we would have $f(a_0) = f(b_0) = f(c_0) = \sigma_2$ and $f(f_0) = \sigma_1$; hence σ_3 has not preimage in $\underline{P}'$.

(3) $f(h_k) \neq \sigma_7$ for every $k \geq 4$.

Suppose $f(h_k) = \sigma_7$ for some $k \geq 4$. By (2) it holds that:
- $f(d_{k-1}) \in \{\sigma_1, \sigma_3\}$;
- $f(d_k) \in \{\sigma_1, \sigma_3\}$.
Let $\delta > h_k$ be such that $f(\delta) = \sigma_4$; necessarily $\delta = e_k$. Hence there is not $\delta > h_k$ such that $f(\delta) = \sigma_5$, and (3) is proved.

(4) If $\alpha \in \mathrm{Dom}(f)$, $f(\alpha) \in \{\sigma_1, \sigma_3\}$.

(5) $f(\gamma) \neq \sigma_7$, $f(\varepsilon) \neq \sigma_7$, $f(\eta) \neq \sigma_7$.

If $f(\gamma) = \sigma_7$, by ($\star$) and (4) there is not $\delta > \gamma$ such that $f(\delta) = \sigma_5$; the same holds for ε. Suppose that $f(\eta) = \sigma_7$ and let $\delta > \eta$ be such that $f(\delta) = \sigma_4$. Necessarily $\delta = \varepsilon$, hence $f(\delta) \neq \sigma_5$ for every $\delta > \eta$.
By (3), (4) and (5), it follows that:

(6) $f(\delta) = \sigma_7$ if and only if $\delta = r_0$

and this proves (i). In particular, f is defined on all the points of $\underline{P}$.

(7) $f(g_k) = \sigma_4$ for every $k \geq 1$.

In fact, $g_k < a_0$, $g_k < f_0$ and, for every $\delta \geq g_k$, $f(\delta) \neq \sigma_5$.
By (2) and (7), we get:

(8) $f(a_k) = f(b_k) = f(c_k) = \sigma_1$ for every $k \geq 0$

and (ii) is proved. Moreover, it follows that:

(9) $f(\alpha) = \sigma_1$, $f(\gamma) = f(\varepsilon) = f(\eta) = \sigma_4$.

(10) There is $n \geq 3$ such that $f(d_n) = \sigma_3$.

If (10) does not hold, then $f(d_k) \neq \sigma_3$ for every $k \geq 3$, hence $f(d_k) = \sigma_1$ for every $k \geq 3$. This implies that σ_3 has not preimage in $\underline{P}$, which is absurd.

(11) $f(d_k) = \sigma_3$ implies $f(d_{k+1}) = \sigma_3$, for every $k \geq 3$.

Suppose $f(d_k) = \sigma_3$; since $f(h_{k+1}) \neq \sigma_7$, necessarily $f(h_{k+1}) = \sigma_5$ and $f(e_{k+1}) = \sigma_5$, therefore $f(d_{k+1}) = \sigma_3$.
From (10) and (11), by induction on $k \geq n$, (iii) follows. Note that we have completely characterized the possible p-morphisms from $\underline{P}$ onto $\underline{P}_{\sigma_7}$. □

By Lemma 5.4.6 and by the fact that $a_{k+1} \sim_j^V d_{k+1}$ for every $k \geq 2$ and $j \leq k$, $\underline{K}$ satisfies Condition (†) of Proposition 5.4.1; therefore:

5.4.7 Proposition $\underline{K}$ *is a model of* $\mathbf{NL}_8^V$. □

5.4.8 Theorem $\mathbf{NL}_8$ *is not strongly ω-complete.*

Proof: Let us consider the following map $g : \underline{P} \rightarrow \underline{P}_{\sigma_7}$.

- $g(a_k) = g(b_k) = g(c_k) = g(\alpha) = \sigma_1$ for every $k \geq 0$.
- $g(f_0) = \sigma_2$.
- $g(d_k) = \sigma_3$ for every $k \geq 3$.
- $g(g_k) = g(\gamma) = g(\varepsilon) = g(\eta) = \sigma_4$ for every $k \geq 1$.
- $g(e_k) = g(h_k) = \sigma_5$ for every $k \geq 4$.
- $g(r_0) = \sigma_7$.

Then g is a p-morphism from $\underline{P}$ onto $\underline{P}_{\sigma_7}$ and, by definition of g, $\underline{P}_{\sigma_7}$ is a V-stable reduction of $\underline{P}$. Since $\underline{K}$ is a V-separable and V-full model of $\mathbf{NL}_8^V$ and $\underline{P}_{\sigma_7}$ is not a frame for $\mathbf{NL}_8$, by the Strong ω-completeness Criterion we can conclude that $\mathbf{NL}_8$ is not strongly ω-complete. □

5.4.3 The logics $\mathbf{NL}_{m+1}$ $(m \geq 9)$ and $\mathbf{NL}_{n+1,n+2}$ $(n \geq 6)$

To treat these logics, firstly we extend the countermodel $\underline{K}$ into the models $\underline{K}_m$ and $\underline{K}_{n,n+1}$, then we proceed as in the previous section. At this aim, we consider a sequence of points $t_1, t_2, \ldots$ defined as follows (see Figure 5.9).

- t_7 coincides with the root r_0 of $\underline{P}$.
- For every $k \neq 7$, $t_k \notin P$.
- For every $k \neq 7$ and every $j \geq 1$, t_j is an immediate successor of t_k if and only if σ_j is an immediate successor of σ_k in $\underline{P}_\omega$ (defined in Section 2.5).

The frames $\underline{P}_m$ and $\underline{P}_{n,n+1}$, for $m \geq 9$ and $n \geq 6$, are defined as follows.

- $\underline{P}_m = \langle P_m, \leq, t_m \rangle$ is the frame having as root t_m.
- $\underline{P}_{n,n+1} = \langle P_{n,n+1}, \leq, r_n \rangle$ is the frame such that:
  - r_n is the root of $\underline{P}_{n,n+1}$;
  - the immediate successors of r_n are t_n and t_{n+1}.

In order to define the models $\underline{K}_m = \langle P_m, \leq, t_m, \Vdash \rangle$ and $\underline{K}_{n,n+1} = \langle P_{n,n+1}, \leq, r_n, \Vdash \rangle$, we consider a suitable increasing sequence of finite sets V_m of propositional variables containing V (used for the logic $\mathbf{NL}_8$) such that the following conditions hold.

(a) a_0, b_0, c_0, f_0 and t_k, for $1 \leq k \leq m$, have V_m-grade 0 in $\underline{K}_m$.

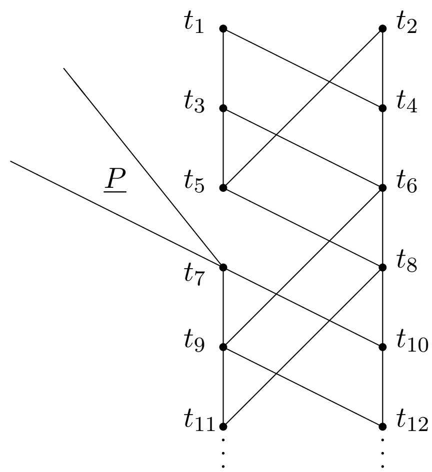

Figure 5.9: The sequence of points t_k

(b) The distinct equivalence classes with respect to the relation $\sim_0^{V_m}$ in $\underline{K}_m$ having more than one element are:
- $\{a_k, b_k, c_k : k \geq 1\} \cup \{d_k : k \geq 3\} \cup \{\alpha\}$
- $\{e_k : k \geq 4\} \cup \{\varepsilon\}$
- $\{g_k : k \geq 1\} \cup \{\gamma\}$
- $\{h_k : k \geq 4\} \cup \{\eta\}$.

(c) a_0, b_0, c_0, f_0, r_n and t_k, for $1 \leq k \leq n+1$, have V_{n+1}-grade 0 in $\underline{K}_{n,n+1}$.

(d) The distinct equivalence classes with respect to the relation $\sim_0^{V_{n+1}}$ in $\underline{K}_{n,n+1}$ having more than one element are as in (b).

5.4.9 Proposition

(i) $\underline{K}_m$ is V_m-separable and V_m-full for every $m \geq 9$.

(ii) $\underline{K}_{n,n+1}$ is V_{n+1}-separable and V_{n+1}-full for every $n \geq 6$.

Proof: We observe that the cone $\underline{P}_{t_7}$ of $\underline{P}_m$ coincides with $\underline{P}$ and all the points t_k have V_m-grade 0; thus (i) follows from Proposition 5.4.5. In a similar way (ii) is also proved. □

5.4.10 Lemma

(i) Let $\underline{P}'$ be a generated subframe of $\underline{P}_m$, with $m \geq 9$, and let f be a p-morphism from $\underline{P}'$ onto $\underline{P}_{\sigma_m}$. Then:
- $f(a_k) = \sigma_1$ for every $k \geq 0$.
- There is $r \geq 3$ such that $f(d_k) = \sigma_3$ for every $k \geq r$.

(ii) Let $\underline{P}'$ be a generated subframe of $\underline{P}_{n,n+1}$, with $n \geq 6$, and let f be a p-morphism from $\underline{P}'$ onto $\underline{P}_{\sigma_{n,n+1}}$. Then:
- $f(a_k) = \sigma_1$ for every $k \geq 0$.
- There is $r \geq 3$ such that $f(d_k) = \sigma_3$ for every $k \geq r$.

Proof:
(i) Suppose that there is a p-morphism f from from some generated subframe $\underline{P}'$ of $\underline{P}_m$ onto $\underline{P}_{\sigma_m}$. Firstly we prove that, for $2 \leq k \leq m$, if f is defined on t_k, then $f(t_k) \leq \sigma_k$. If $2 \leq k \leq 6$ or $k = 8$ the proof is immediate. Suppose, by absurd, that $f(t_7) > \sigma_7$; if $f(t_7) \geq \sigma_9$, then there must be $\delta > t_7$ such that $f(\delta) = \sigma_7$, in contradiction with Point (i) of Lemma 5.4.6. Suppose that $f(t_7) = \sigma_8$ and let δ be such that $f(\delta) = \sigma_7$. It is not the case that $\delta < t_7$ or $\delta > t_8$; this implies that $\delta = \sigma_8$, but one can easily check that this is not possible; we can conclude that $f(t_7) \leq \sigma_7$. If $9 \leq k \leq m$, then the immediate successors of t_k are t_{k-2} and t_{k-3}; if f is defined on t_k, then $f(t_{k-2}) \leq \sigma_{k-2}$ and $f(t_{k-3}) \leq \sigma_{k-3}$, therefore $f(t_k) \leq \sigma_k$. On the other hand there must be a point δ of $\underline{P}_{\sigma_m}$ such that $f(\delta) = \sigma_m$; it follows that $\delta = t_m$, hence, for all $1 \leq k \leq m-2$, $f(t_k) = \sigma_k$. In particular, $f(t_7) = \sigma_7$ and, by Lemma 5.4.6, (i) follows.
(ii) Observe that f cannot be defined on r_n. Arguing as in (i), we can prove that $f(t_k) = \sigma_k$ for $1 \leq k \leq m+1$; hence $f(t_7) = \sigma_7$ and, by Lemma 5.4.6, (ii) is proved.
□

It follows that the models $\underline{K}_m$ and $\underline{K}_{n,n+1}$ satisfy Conditions (†) and (††) of Proposition 5.4.1 respectively, therefore:

5.4.11 Proposition
(i) $\underline{K}_m$ is a model of $\mathbf{NL}^{V_m}_{m+1}$ for every $m \geq 9$.

(ii) $\underline{K}_{n,n+1}$ is a model of $\mathbf{NL}^{V_{n+1}}_{n+1,n+2}$ for every $n \geq 6$.

□

5.4.12 Theorem *The logics $\mathbf{NL}_{m+1}$ and $\mathbf{NL}_{n+1,n+2}$, for every $m \geq 9$ and $n \geq 6$, are not strongly ω-complete.*

Proof: Let g be the p-morphism from $\underline{P}$ onto $\underline{P}_{\sigma_7}$ defined in Theorem 5.4.8; we extend g to $\underline{P}_m$ as follows:

- $g(t_k) = \sigma_k$ for every $1 \leq k \leq m$.

By definition of g, we can assert that $\underline{P}_{\sigma_m}$ is a V_m-stable reduction of $\underline{P}_m$; since $\underline{K}_m$ is a V_m-separable and V_m-full model of $\mathbf{NL}^{V_m}_{m+1}$ and $\underline{P}_{\sigma_m}$ is not a frame for $\mathbf{NL}_{m+1}$, by the Strong ω-completeness Criterion it follows that $\mathbf{NL}_{m+1}$ is not strongly ω-complete.
Let $\underline{P}_{\tilde{\sigma}_n}$ be the frame obtained by adding to $\underline{P}_{\sigma_{n,n+1}}$ a root $\tilde{\sigma}_n$. We extend g to $\underline{P}_{n,n+1}$ as follows:

- $g(t_k) = \sigma_k$ for every $1 \leq k \leq n+1$;

- $g(r_n) = \tilde{\sigma}_n$.

By definition of g, $\underline{P}_{\tilde{\sigma}_n}$ is a V_{n+1}-stable reduction of $\underline{P}_{n,n+1}$; since $\underline{K}_{n,n+1}$ is a V_{n+1}-separable and V_{n+1}-full model of $\mathbf{NL}^{V_{n+1}}_{n+1,n+2}$ and $\underline{P}_{\tilde{\sigma}_n}$ is not a frame for $\mathbf{NL}_{n+1,n+2}$, the logic $\mathbf{NL}_{n+1,n+2}$ is not strongly ω-complete. □

5.4.4 The logic $\mathbf{NL}_9$

It remains to analyze the logic $\mathbf{NL}_9$. To treat this case, we define the frame $\underline{P}_8 = \langle P_8, \leq, z_0\rangle$ of root z_0 and the model $\underline{K}_8 = \langle P_8, \leq, z_0, \Vdash\rangle$. The points a_k, b_k, c_k, with $k \geq 0$, d_j, e_j, with $j \geq 3$, α, ε, f_0 are defined as in Section 5.4.2. We introduce new sequences of points $l_3, l_4, \ldots, \lambda$, $m_4, m_5, \ldots, \mu$ and $n_4, n_5, \ldots, \nu$ in the following way.

- The immediate successors of l_k are a_{k-1}, c_{k-2} and f_0 for every $k \geq 3$.

- $\alpha \not\leq l_k$ and $\lambda \not\leq l_k$ for every $k \geq 3$.

(See Figure 5.10.)

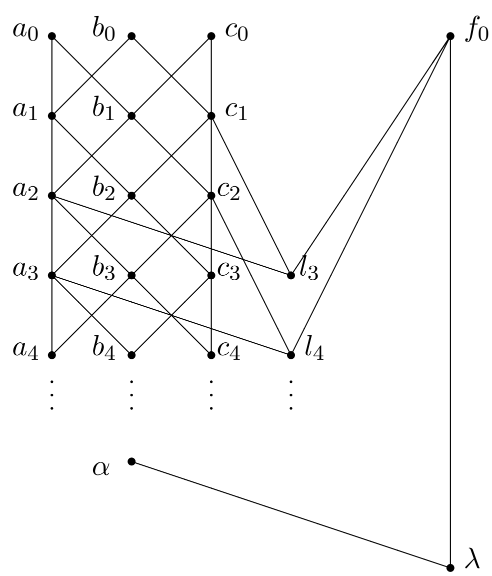

Figure 5.10: The sequence of points l_k

- The immediate successors of m_k are d_{k-1}, l_{k-1} and l_k for every $k \geq 4$

- The only immediate successor of μ is λ.

- $\mu \not\leq d_k$ and $\mu \not\leq l_k$ for every $k \geq 3$.

(See Figure 5.11.)

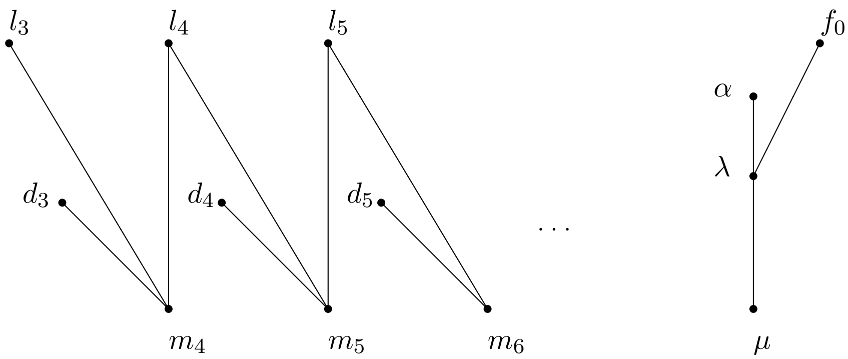

Figure 5.11: The sequence of points m_k

- The immediate successors of n_k are e_{k-1} and e_k for every $k \geq 4$.
- The only immediate successor of ν is ε.
- $\nu \not\leq d_k$ and $\nu \not\leq l_k$ for every $k \geq 3$.

(See Figure 5.12.) The immediate successors of the root z_0 are the points m_k and

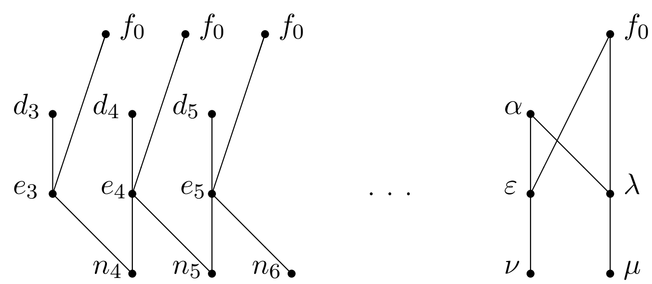

Figure 5.12: The sequences of points e_k and n_k

n_k, for every $k \geq 4$, μ and ν.
The forcing relation is defined in such a way that the following properties hold with respect to some finite set of propositional variables W.

- a_0, b_0, c_0, f_0, z_0 have W-grade 0.
- The distinct equivalence classes with respect to the relation $\sim_0^W$ having more than one element are:
  - $\{a_k, b_k, c_k : k \geq 1\} \cup \{d_k : k \geq 3\} \cup \{\alpha\}$
  - $\{e_k : k \geq 3\} \cup \{\varepsilon\}$
  - $\{l_k : k \geq 3\} \cup \{\lambda\}$
  - $\{m_k : k \geq 4\} \cup \{\mu\}$
  - $\{n_k : k \geq 4\} \cup \{\nu\}$.

The proof goes on according to the lines of the previous sections.

5.4.13 Lemma *For every $r \geq 0$ the following holds.*

(i) $a_r, b_r, c_r, d_r, e_r, l_r, m_r, n_r$ *(when defined) have W-grade r.*

(ii) The distinct equivalence classes with respect to the relation $\sim_r^W$ having more than one element are:
- $\{a_k, b_k, c_k, d_k : k \geq r+1\} \cup \{\alpha\}$
- $\{e_k : k \geq r+1\} \cup \{\varepsilon\}$
- $\{l_k : k \geq r+1\} \cup \{\lambda\}$
- $\{m_k : k \geq r+1\} \cup \{\mu\}$
- $\{n_k : k \geq r+1\} \cup \{\nu\}$.

(iii) $\alpha, \varepsilon, \lambda, \mu, \nu$ *have infinite W-grade.*

□

It follows that:

5.4.14 Proposition $\underline{K}_8$ *is W-separable and W-full.* □

Now, we study the p-morphisms on $\underline{P}_8$.

5.4.15 Lemma *Let $\underline{P}'$ be a generated subframe of $\underline{P}_8$ and let f be a p-morphism from $\underline{P}'$ onto $\underline{P}_{\sigma_8}$. Then:*

(i) $f(a_k) = \sigma_1$ *for every $k \geq 0$.*

(ii) One of the following facts (A) and (B) holds.
(A) $f(\alpha) = \sigma_3$.
(B) There is $r \geq 3$ such that $f(d_k) = \sigma_3$ for every $k \geq r$.

Proof: Let $\underline{P}'$ be a generated subframe of $\underline{P}_8$ such that there is a p-morphism f from $\underline{P}'$ onto $\underline{P}_{\sigma_8}$; we proceed as in in the proof of Lemma 5.4.6. We firstly observe that $\{a_0, b_0, c_0, f_0\} \subseteq \mathrm{Dom}(f)$, hence $\{f(a_0), f(b_0), f(c_0), f(f_0)\} \subseteq \{\sigma_1, \sigma_2\}$.

(1) $f(a_0) = f(b_0) = f(c_0)$.

Otherwise:

- $\{a_2, b_2, c_2\} \subseteq \mathrm{Dom}(f)$ and $\{f(a_2), f(b_2), f(c_2)\} \subseteq \{\sigma_4, \sigma_6\}$.

This implies that σ_5 has not preimage in $\underline{P}'$, in fact:

- if $\delta \in \{a_0, b_0, c_0, a_1, b_1, c_1, f_0\}$, $f(\delta) \neq \sigma_5$;
- in the remaining cases, either $\delta \leq a_2$ or $\delta \leq b_2$ or $\delta \leq c_2$, hence $f(\delta) \neq \sigma_5$.

It follows that:

(2) $f(a_0) = f(b_0) = f(c_0) = \sigma_1$ and $f(f_0) = \sigma_2$.

By (2), we can easily prove that:

(3) $f(\delta) = \sigma_8$ if and only if $\delta = z_0$.

Therefore f is defined on all the points of $\underline{P}_8$.

(4) $f(l_3) = \sigma_4$.

If $f(l_3) \neq \sigma_4$, necessarily $f(l_3) = \sigma_5$, hence $f(\delta^*) = \sigma_3$ for some $\delta^* \in \{a_2, a_1, b_1, c_1\}$. Let $\delta > z_0$ be such that $f(\delta) = \sigma_4$; since $\delta < f_0$, it follows that $\delta < \delta^*$, which is absurd.

(5) $f(l_k) = \sigma_4$ implies $f(l_{k+1}) = \sigma_4$, for every $k \geq 3$.

If $f(l_k) = \sigma_4$, then either $f(m_{k+1}) = \sigma_4$ or $f(m_{k+1}) = \sigma_6$, therefore $f(l_{k+1}) = \sigma_4$. By (4) and (5), we can infer that:

(6) $f(l_k) = \sigma_4$ for every $k \geq 3$.

This implies that:

(7) $f(a_k) = f(b_k) = f(c_k) = \sigma_1$ for every $k \geq 0$.

Thus (i) is proved. To prove (ii), suppose that $f(\alpha) \neq \sigma_3$; we show that Condition (B) holds. We can assert:

(8) $f(\alpha) = \sigma_1$, $f(\varepsilon) = f(\lambda) = f(\mu) = f(\nu) = \sigma_4$.

(9) There is $r \geq 3$ such that $f(e_r) = \sigma_5$.

Otherwise, $f(e_k) \neq \sigma_5$ for every $k \geq 3$, that is $f(e_k) = \sigma_4$ for every $k \geq 3$. It follows that σ_5 has not preimage in $\underline{P}_8$.

(10) $f(e_k) = \sigma_5$ implies $f(e_{k+1}) = \sigma_5$, for every $k \geq 3$.

In fact, if $f(e_k) = \sigma_5$, necessarily $f(n_{k+1}) = \sigma_5$ and $f(e_{k+1}) = \sigma_5$. Therefore:

(11) There is $r \geq 3$ such that $f(e_k) = \sigma_5$ for every $k \geq r$.

Since $f(e_k) = \sigma_5$ implies $f(d_k) = \sigma_3$, (B) is proved. □

5.4.16 Proposition *$\underline{K}_8$ is a model of* $\mathbf{NL}_9^W$.

Proof: We prove that $\underline{K}_8$ satisfies Condition (†) of Proposition 5.4.1. Let f be a p-morphism from some generated subframe $\underline{P}'$ of $\underline{P}_8$ onto $\underline{P}_{\sigma_8}$ and let $k \geq 0$. If Condition (A) of Lemma 5.4.15 holds, then a_{k+1} and α satisfy (†) of Proposition 5.4.1. Otherwise, let $r \geq 3$ be as in (B) and let $m = \max(k, r)+1$; we have that $a_m \sim_k^W d_m$, $f(a_m) = \sigma_1$ and $f(d_m) = \sigma_3$, thus (†) of Proposition 5.4.1 holds also in this case. □

5.4.17 Theorem $\mathbf{NL}_9$ *is not strongly ω-complete.*

Proof: Let us consider the following map $g : \underline{P}_8 \to \underline{P}_{\sigma_8}$.

- $g(a_k) = g(b_k) = g(c_k) = g(\alpha) = \sigma_1$ for every $k \geq 0$.
- $g(f_0) = \sigma_2$.
- $g(d_k) = \sigma_3$ for every $k \geq 3$.
- $g(l_k) = g(\varepsilon) = g(\lambda) = g(\mu) = g(\nu) = \sigma_4$ for every $k \geq 3$.
- $g(e_k) = g(n_j) = \sigma_5$ for every $k \geq 3$ and every $j \geq 4$.
- $g(m_k) = \sigma_6$ for every $k \geq 4$.
- $g(z_0) = \sigma_8$.

Then g is a p-morphism from $\underline{P}_8$ onto $\underline{P}_{\sigma_8}$ and, by definition of g, $\underline{P}_{\sigma_8}$ is a W-stable reduction of $\underline{P}_8$. Since $\underline{K}_8$ is a W-separable and W-full model of $\mathbf{NL}_9^W$ and $\underline{P}_{\sigma_8}$ is not a frame for $\mathbf{NL}_9$, by the Strong ω-completeness Criterion we can conclude that $\mathbf{NL}_9$ is not strongly ω-complete. □

5.4.5 Non extensive ω-canonicity of $\mathbf{NL}_{5,6}$

To complete the survey on the logics in one variable, we prove that the logic $\mathbf{NL}_{5,6}$ is not extensively ω-canonical by exhibiting, for some finite V, a well V-separable model $\underline{K} = \langle P, \leq, w_0, \Vdash \rangle$ of $\mathbf{NL}_{5,6}^V$ based on a a frame $\underline{P} = \langle P, \leq, w_0 \rangle$ which is not a frame for such a logic. The points a_k, b_k, c_k, d_j, for $k \geq 0$ and $j \geq 3$, are defined as in Section 5.4.2; the points u_0, w_0 and v_k, for $k \geq 4$, are defined as follows (see Figure 5.13):

- $u_0 < \delta$ iff $\delta = a_k$ or $\delta = b_k$ or $\delta = c_k$, for some $k \geq 0$, or $\delta = f_0$.
- The immediate successors of v_k are v_{k+1}, d_{k-1} and f_0 for every $k \geq 4$.
- The immediate successors of w_0 are u_0 and v_4.

The forcing relation is defined in such a way that the following properties hold with respect to some finite set V of propositional variables.

- $a_0, b_0, c_0, f_0, u_0, w_0$ have V-grade 0.
- The distinct equivalence classes with respect to the relation $\sim_0^V$ having more than one element are:
  - $\{a_k, b_k, c_k : k \geq 1\} \cup \{d_k : k \geq 3\}$
  - $\{v_k : k \geq 4\}$.

Thus we have:

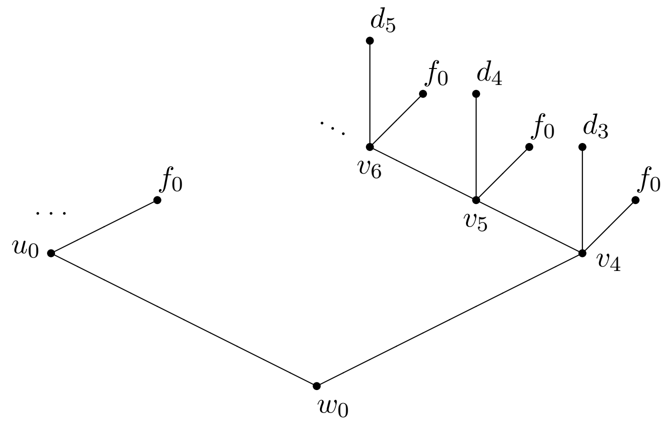

Figure 5.13: The model $\underline{K}$ of $\mathbf{NL}_{5,6}$

- a_n, b_n, c_n, d_n, v_n (when defined) have V-grade n.
- The distinct equivalence classes with respect to the relation $\sim_n^V$ having more than one element are:
  - $\{a_k, b_k, c_k, d_k : k \geq n+1\}$
  - $\{v_k : k \geq n+1\}$.

Since all the points of $\underline{K}$ have finite V-grade, the following fact is immediate.

5.4.18 Proposition *$\underline{K}$ is a well V-separable Kripke model.* □

Of course, $\underline{K}$ cannot be V-full, since there are no points of infinite V-grade. In order to obtain a V-full equivalent model, we have to add two points α and υ which are the limits of the non definitively constant V-sequences contained in the sets $\{a_k, b_k, c_k, d_j : k \geq 0, j \geq 3\}$ and $\{v_k : k \geq 4\}$ respectively.

5.4.19 Lemma *Let $\underline{P}'$ be a generated subframe of $\underline{P}$ and let f be a p-morphism from $\underline{P}'$ onto $\underline{P}_{\sigma_{4,5}}$. Then:*

- *$f(a_k) = \sigma_1$ for every $k \geq 0$;*
- *$f(d_k) = \sigma_3$ for infinitely many k.*

Proof: Let $\underline{P}'$ be a generated subframe of $\underline{P}$ such that there is a p-morphism f from $\underline{P}'$ onto $\underline{P}_{\sigma_{4,5}}$ (clearly $w_0 \notin \mathrm{Dom}(f)$). As in the proof of Lemma 5.4.6, we have:

(1) $f(a_0) = f(b_0) = f(c_0) = \sigma_1$ and $f(f_0) = \sigma_2$.

(2) $f(u_0) = \sigma_4$.

If $f(u_0) \neq \sigma_4$, necessarily $f(v_n) = \sigma_4$ for some $n \geq 4$. It follows that $f(\delta) = \sigma_3$ iff $\delta = d_k$ for some $3 \leq k \leq n-2$, hence σ_5 has not preimage in $\underline{P}'$, which is absurd. Therefore (2) holds and, as an immediate consequence, we have:

(3) $f(a_k) = f(b_k) = f(c_k) = \sigma_1$ for every $k \geq 0$.

Let $n \geq 4$ be such that $f(v_n) = \sigma_5$; then:

(4) $f(v_k) = \sigma_5$ for every $k \geq n$.

This allows us to infer that:

(5) For every $k \geq 3$ there is $m \geq k$ such that $f(d_m) = \sigma_3$.

Thus the lemma is proved. □

We can now state the main result of this section.

5.4.20 Theorem *The logic* $\mathbf{NL}_{5,6}$ *is not extensively ω-canonical.*

Proof: By Proposition 5.4.18, $\underline{K}$ is well V-separable; by Lemma 5.4.19, $\underline{K}$ satisfies Condition (††) of Proposition 5.4.1, therefore $\underline{K}$ is a model of $\mathbf{NL}^V_{5,6}$. On the other hand, it is easy to define a p-morphism from some generated subframe $\underline{P}'$ of $\underline{P}$ onto $\underline{P}_{\sigma_{4,5}}$; hence $\underline{P}$ is not a frame for $\mathbf{NL}_{5,6}$ and $\mathbf{NL}_{5,6}$ is not extensively ω-canonical. □

5.5 Some conclusions

The reader can notice that there is a close correspondence (at least at an high level) between the techniques used in the context of ω-canonicity and the ones used for canonicity. One may wonder whether there are advantages in extend the techniques used for ω-canonicity to the case of canonicity. More precisely, instead of using the chains to build countermodels for strong completeness, we could directly define a model $\underline{K}$ and then study the separability, the fullness and the other properties by defining suitable notions of "grade" of a point. At first glance, this approach seems to be more complex, and we do not know whether we can get significative improvements.

Another interesting question is the extension of these methods to modal logics. For instance, in [15] it is shown that the canonicity and strong completeness criteria can be easily reformulated for modal logics and, as an example, a new proof of the non strong completeness of the modal logic **K.1** (obtained by adding to the normal modal system **K** the axiom scheme $\Box\Diamond p \to \Diamond\Box p$) is given. Our impression is that there are not real difficulties in transpose these techniques in a modal framework.

Appendix A

Further properties on ω-canonical models

In this section we will show some interesting properties of ω-canonical models (that is, V-separable and V-full models, with V finite). The material of this section is still unpublished; our aim is to develop the ideas here explained in order to obtain filtration techniques, directly applied to ω-canonical models, more powerful than the existing ones. For instance, if one attempts to prove the finite model property of **RH** (that is to say, the decidability of such a logic) the known filtration techniques are not suited.
Let V be a finite set of propositional variables and let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a V-separable and V-full model. We say that a point α of $\underline{K}$ is *finite* if the cone $\underline{P}_\alpha$ is finite; we say that α is *infinite* if it is not finite. We recall that the finite points of a V-separable model $\underline{K}$ (with V finite) are exactly the points of finite depth. The following result (see [7]) shows that the finite points of $\underline{K}$ are well characterized by means of V-formulas.

A.0.1 Proposition *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be V-separable and V-full model, where V is a finite set of propositional variables, and let α be a finite point of $\underline{K}$. Then, there are two V-formulas H_α and G_α such that, for every $\beta \in P$, it holds that:*

(i) $\beta \Vdash H_\alpha$ *if and only if* $\alpha \leq \beta$.

(ii) $\beta \Vdash G_\alpha$ *if and only if* $\beta \not\leq \alpha$.

□

We say that a point α of $\underline{K}$ is *infinite maximal* if α is infinite and every $\beta > \alpha$ is finite. A remarkable property of ω-canonical models is that every infinite point sees at least an infinite maximal point, as proved in next proposition.

A.0.2 Proposition *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a V-separable and V-full model (where V is finite), and let α be an infinite point of $\underline{K}$. Then there is a point β of $\underline{K}$ such that $\alpha \leq \beta$ and β is infinite maximal.*

Proof: Let $\mathscr{I}$ be the set

$$\mathscr{I} = \{\beta : \alpha \leq \beta \text{ and } \beta \text{ is infinite}\}.$$

Let $\mathscr{C} \subseteq \mathscr{I}$ be a chain of elements of $\mathscr{I}$, that is, $\mathscr{C}$ is a subset of $\mathscr{I}$ whose elements are totally ordered by the partial ordering relation $\leq$. Let us define the set:

$$\Delta = \bigcup_{\varepsilon \in \mathscr{C}} \Gamma^V_{\underline{K}}(\varepsilon).$$

It is not difficult to prove that Δ is a V-saturated set and that $\Gamma^V_{\underline{K}}(\alpha) \subseteq \Delta$. By the V-fullness of $\underline{K}$, there is $\delta \in P$ such that $\alpha \leq \delta$ and $\Gamma^V_{\underline{K}}(\delta) = \Delta$; moreover, by the well V-separability of $\underline{K}$, $\varepsilon \leq \delta$, for all $\varepsilon \in \mathscr{C}$. We now prove that δ is infinite. Suppose that δ is finite and let H_δ be the V-formula defined in Proposition A.0.1. We have that $\delta \Vdash H_\delta$; thus, by definition of Δ, $\varepsilon \Vdash H_\delta$, for some $\varepsilon \in \mathscr{C}$. It follows that $\delta \leq \varepsilon$, which is absurd, since ε is infinite and δ finite; thus δ must be infinite. Therefore we have proved that δ is an element of $\mathscr{I}$ which is an upper bound (with respect to $\leq$) of $\mathscr{C}$. We can apply Zorn Lemma (see [21]) and claim that $\mathscr{I}$ has a maximal element β^* (with respect to $\leq$). It is immediate to see that β^* is an infinite maximal point. □

A.0.3 Proposition *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a V-separable and V-full model (where V is finite), let α be an infinite maximal point of $\underline{K}$. Then α has no immediate successors.*

Proof: Suppose, by absurd, that α has an immediate successor β. We firstly prove the following fact:

(A) For every δ such that $\alpha < \delta$, there is an immediate successor δ^* of α such that $\delta^* \leq \delta$.

Let δ be such that $\alpha < \delta$; if $\beta \leq \delta$ then (A) is already proved; otherwise, let us consider the set

$$\mathscr{S} = \{\delta' : \alpha < \delta' \leq \delta\}.$$

Reasoning as in the previous proposition (using the formula G_β), we can show that $\mathscr{S}$ has a minimal element δ^*, and this completes the proof of (A).
Let $\beta_1, \beta_2, \ldots, \beta_n, \ldots$ be an enumeration of all the immediate successors of α (we point out that the set of immediate successors of α is actually numerable, since it is a subset of the numerable set of all the finite points of $\underline{K}$). We prove that:

(B) $\Gamma^V_{\underline{K}}(\alpha) \cup \{G_{\beta_1}\} \nvdash_{INT} \{H_{\beta_1}, H_{\beta_2}, \ldots, H_{\beta_n}, \ldots\}$

where $G_{\beta_1}, H_{\beta_1}, \ldots, H_{\beta_n}, \ldots$ are the V-formulas defined in Proposition A.0.1. Suppose that (B) does not hold. Then, for some $A_1, \ldots, A_k$ in $\Gamma^V_{\underline{K}}(\alpha)$ and some integers $j_1, \ldots, j_m$, we have that:

(*) $A_1 \wedge \cdots \wedge A_k \wedge\ G_{\beta_1} \vdash_{INT} H_{\beta_{j_1}} \vee \cdots \vee H_{\beta j_m}$.

Let $l = max\{j_1, \ldots, j_m\} + 1$. Then:

- $\beta_l \Vdash A_1, \ldots, \beta_l \Vdash A_k$ (in fact, $\alpha < \beta_l$);
- $\beta_l \Vdash G_{\beta_1}$ (in fact, $\beta_l \not\leq \beta_1$).

By (*), it follows that $\beta_l \Vdash H_{\beta_s}$, for some $s \in \{j_1, \ldots, j_m\}$, hence $\beta_s \leq \beta_l$, which implies that $l = s$, a contradiction; thus (B) is proved. We can apply the Inclusion-exclusion Lemma and claim that there exists a V-saturated set Δ such that:

- $\Gamma^V_{\underline{K}}(\alpha) \subseteq \Delta$;
- $G_{\beta_1} \in \Delta$;
- for every $n \geq 1$, $H_{\beta_n} \notin \Delta$.

Let, by the V-fullness of $\underline{K}$, $\delta \in P$ be such that $\alpha \leq \delta$ and $\Gamma^V_{\underline{K}}(\delta) = \Delta$. Since $\alpha \nVdash G_{\beta_1}$, then $\alpha \neq \delta$; by (A), there is $n \geq 1$ such that $\beta_n \leq \delta$. This implies that $\delta \Vdash H_{\beta_n}$, that is $H_{\beta_n} \in \Delta$, a contradiction. We can conclude that the initial hypothesis is false and α has no immediate successors. □

The following proposition about infinite maximal points of ω-canonical models is the main result of this section.

A.0.4 Proposition *Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a V-separable and V-full model (V finite), and let α be an infinite maximal point of $\underline{K}$. Then the cone $\underline{P}_\alpha$ of $\underline{P} = \langle P, \leq \rangle$ has the filter property.*

Proof: Suppose, by absurd, that $\underline{P}_\alpha$ has not the filter property; then there are β_1 and β_2 in $\underline{P}$ such that:

(1) $\alpha < \beta_1$ and $\alpha < \beta_2$;

(2) for every $\delta \in P$, if $\alpha < \delta \leq \beta_1$, then $\delta \not\leq \beta_2$.

Reasoning as before (and using the formula G_{β_2}), we can prove that the nonempty set

$$\mathscr{S} = \{\delta : \alpha < \delta \leq \beta_1\}$$

has a minimal element β^*. It follows that β^* is an immediate successor of α, in contradiction with the previous proposition. Thus $\underline{P}_\alpha$ has the filter property. □

In this way we have completely described the structure of an ω-canonical model up to the "wall" of infinite maximal points. The question is whether there other natural walls of this kind; we think that a knowledge of these barriers could lead to significative improvements in filtration methods.

Finally, as an immediate consequence of the previous proposition, and taking into account Proposition 1.9.6 and Proposition 1.3.1, we have:

A.0.5 Corollary *Let L be an intermediate logic, let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a V-separable and V-full model of L^V (V finite), and let α be an infinite maximal point of $\underline{K}$. Then the cone $\underline{P}_\alpha$ of $\underline{P} = \langle P, \leq \rangle$ is a frame for L.* □

One can verify these properties by examining the numerous examples of ω-canonical models contained in Chapter 5.

Appendix B

An overview of the main results

We give a summarizing picture of the main notations and results.

B.1 Intermediate logics

Schema Axioms		
$\boldsymbol{bd}_1$	$=$	$p_1 \vee \neg p_1$
$\boldsymbol{bd}_{n+1}$	$=$	$p_{n+1} \vee (p_{n+1} \rightarrow \boldsymbol{bd}_n) \quad n \geq 1$
$\boldsymbol{bb}_n$	$=$	$\bigwedge_{i=0}^{n}((p_i \rightarrow \bigvee_{j \neq i} p_j) \rightarrow \bigvee_{j \neq i} p_j) \rightarrow \bigvee_{i=0}^{n} p_i \quad n \geq 2$
$\boldsymbol{dum}$	$=$	$(p \rightarrow q) \vee (q \rightarrow p)$
$\boldsymbol{kp}$	$=$	$(\neg p \rightarrow q_1 \vee q_2) \rightarrow (\neg p \rightarrow q_1) \vee (\neg p \rightarrow q_2)$
$\boldsymbol{nf}_1$	$=$	p
$\boldsymbol{nf}_2$	$=$	$\neg p$
$\boldsymbol{nf}_3$	$=$	$\neg\neg p$
$\boldsymbol{nf}_4$	$=$	$\neg\neg p \rightarrow p$
$\boldsymbol{nf}_k$	$=$	$\boldsymbol{nf}_{k-1} \rightarrow \boldsymbol{nf}_{k-3} \vee \boldsymbol{nf}_{k-4} \quad k \geq 5$

Intermediate logics		
$\mathbf{Bd}_n$	=	$\mathbf{Int} + \boldsymbol{bd}_n$ $(n \geq 1)$
LC	=	$\mathbf{Int} + \boldsymbol{dum}$
KP	=	$\mathbf{Int} + \boldsymbol{kp}$
$\mathbf{T}_n$	=	$\mathbf{Int} + \boldsymbol{bb}_n$ $(n \geq 2)$
MV	=	**Int + ?**
RH	=	**Int + ?**

Remark: For **MV** and **RH** only semantical characterizations are known.

Logics in one variable		
Cl	=	$\mathbf{Int} + (\boldsymbol{nf}_1 \vee \boldsymbol{nf}_2) = \mathbf{Int} + \boldsymbol{nf}_4$
Jn	=	$\mathbf{Int} + (\boldsymbol{nf}_2 \vee \boldsymbol{nf}_3) = \mathbf{Int} + \boldsymbol{nf}_5$
$\mathbf{NL}_n$	=	$\mathbf{Int} + \boldsymbol{nf}_n$ $\quad n \geq 6$
$\mathbf{NL}_{m,m+1}$	=	$\mathbf{Int} + \boldsymbol{nf}_{m,m+1}$ $\quad m \geq 3$
St	=	$\mathbf{NL}_6$
Ast	=	$\mathbf{NL}_7$

B.2 Classification of logics

Classification of some intermediate logics			
$\mathbf{Bd}_n$	$(n \geq 1)$	hypercanonical ($QHYP_0$)	
LC		hypercanonical ($QHYP_1$)	non $QHYP_2$
Jn		hypercanonical ($QHYP_2$)	non $QHYP_1$
KP		canonical	non extensively canonical non extensively ω-canonical
$\mathbf{T}_n$	$(n \geq 2)$	non strongly ω-complete	
MV		non extensively canonical	canonical **?**
RH		non canonical	strongly complete **?**

Classification of the logics in one variable		
Canonical:		
Cl	hypercanonical	$QHYP_0$
Jn	hypercanonical	$QHYP_2$, non $QHYP_1$
$\mathbf{NL}_{3,4}$	hypercanonical	$QHYP_1$ and $QHYP_2$
$\mathbf{NL}_{4,5}$	extensively canonical	non hypercanonical
ω-canonical and Non strongly complete:		
St	extensively ω-canonical	
$\mathbf{NL}_{5,6}$	non extensively ω-canonical	
Ast	non extensively ω-canonical	
$\mathbf{NL}_{6,7}$	non extensively ω-canonical	
Non strongly ω-complete:		
$\mathbf{NL}_{n,n+1}$	with $n \geq 7$	
$\mathbf{NL}_m$	with $m \geq 8$	

Classification of the logics in one variable with bounded depth		
$\mathbf{Jn} + \boldsymbol{bd}_h$	hypercanonical	for $h \geq 1$
$\mathbf{NL}_{3,4} + \boldsymbol{bd}_h$	hypercanonical	for $h \geq 1$
$\mathbf{NL}_{4,5} + \boldsymbol{bd}_h$	extensively canonical	for $h \geq 1$
$\mathbf{NL}_7 + \boldsymbol{bd}_h = \mathbf{Ast} + \boldsymbol{bd}_h$	hypercanonical	for $1 \leq h < 4$
	canonical non extensively canonical	for $h \geq 4$
	extensively ω-canonical	for $h \geq 1$
	$m \geq 6$ and $m \neq 7$ $\quad h_m = m$ Div $2 + 1$	
$\mathbf{NL}_m + \boldsymbol{bd}_h$	hypercanonical	for $1 \leq h < h_m$
	non strongly complete	for $h \geq h_m$
	extensively ω-canonical	for $h \geq 1$
	$m \geq 5$ $\quad k_m = (m+3)$ Div $2 + 1$	
$\mathbf{NL}_{m,m+1} + \boldsymbol{bd}_h$	hypercanonical	for $1 \leq h < k_m$
	non strongly complete	for $h \geq k_m$
	extensively ω-canonical	for $h \geq 1$

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